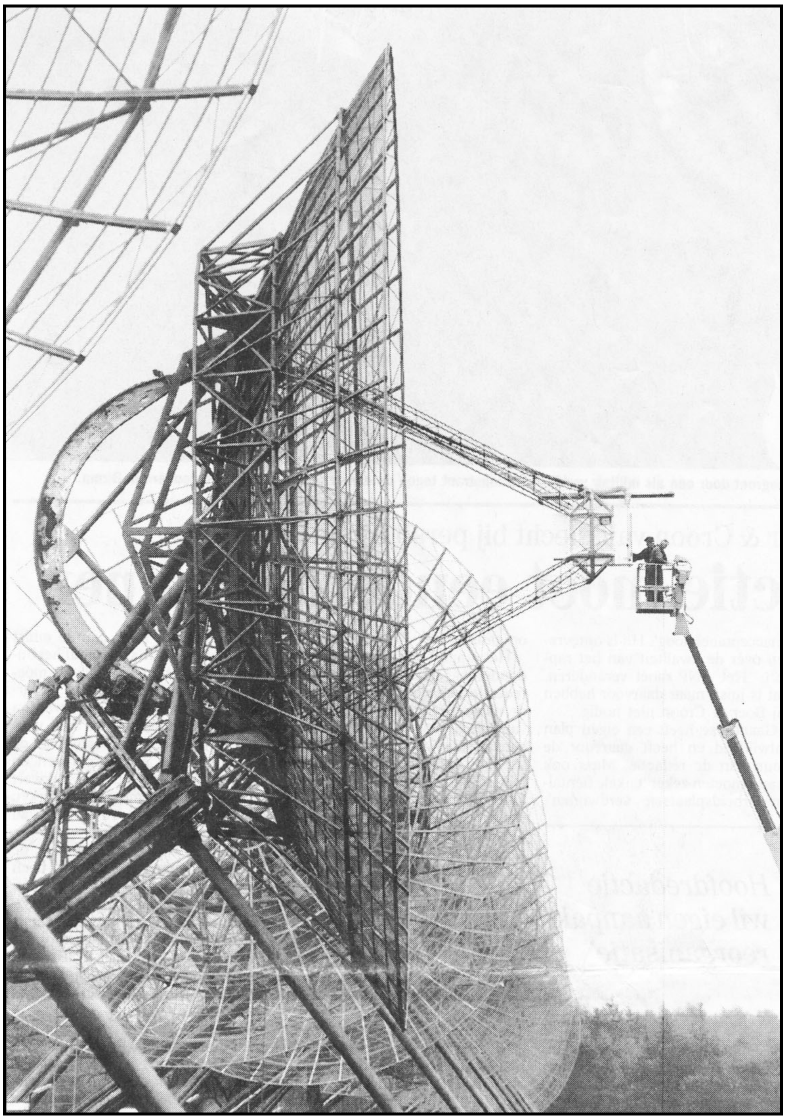
Conflict lines and Reflections



Advanced geometry for senior highschool
Profile Nature &Technology
Freudenthal Institute

Conflict lines and Reflections; advanced geometry, part 3

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Profile: Nature and Technology
Class: VWO 6
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Preface

edge and conflict

Throughout history differences of opinions about the exact course of a border often led to conflicts and even wars. In the first two sections of chapter one we will take a look at several division- and border problems:

– To whom does the oil and gas which are found at the bottom of the North Sea belong?
– How can you divide 'new land' among old neighbors?
– How do you determine the 'middle' of a border river, if that is exactly where the edge should be?

We are looking for peaceful mathematical solutions, of course.

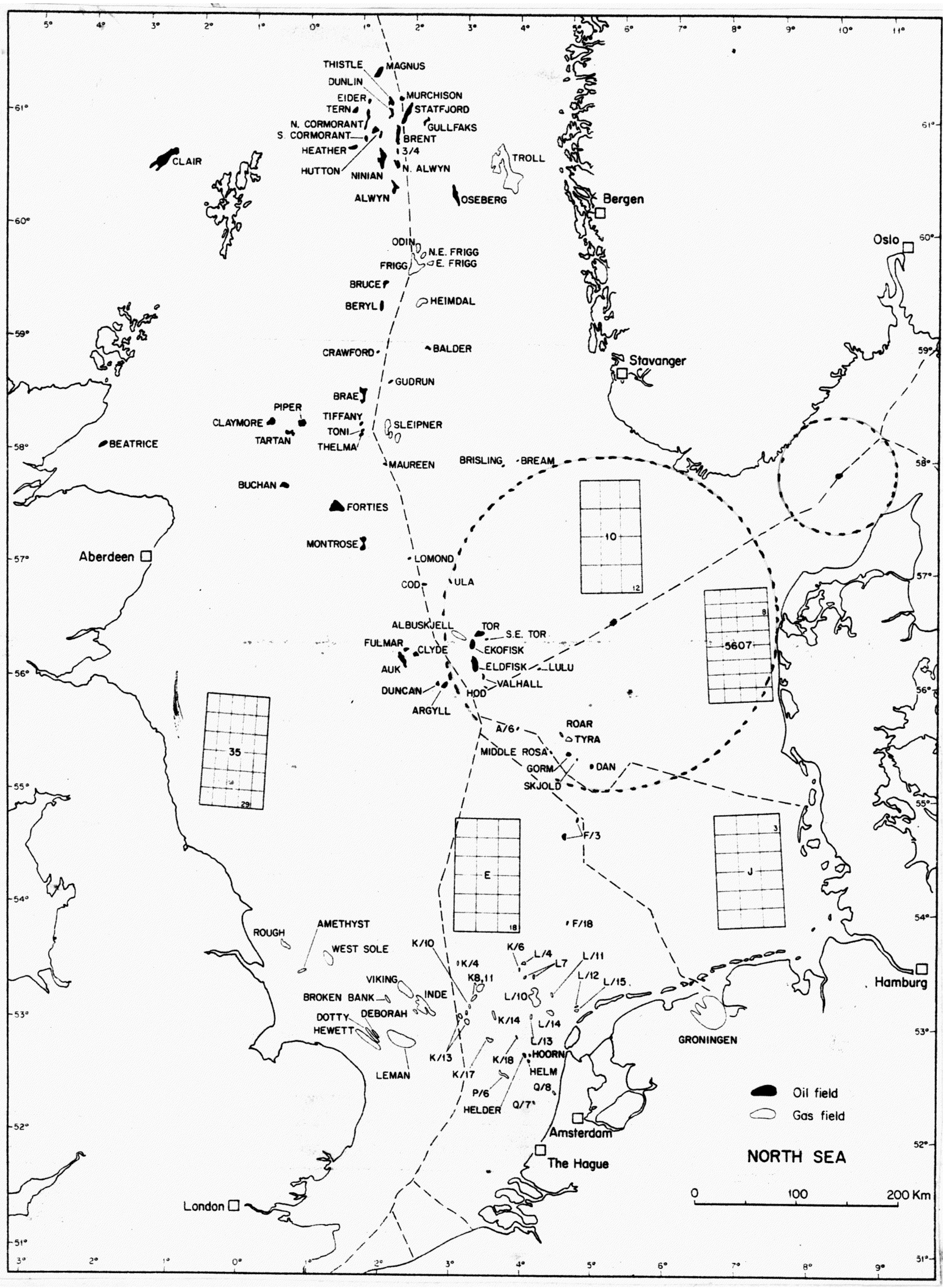
parabola
ellipse
hyperbola

A systematic research of a special type of borderlines (which we will call conflict lines) brings us to classic curves, namely *parabolas, ellipses* and *hyperbolas*. In this research we will again use CABRI on the computer.

The special properties of these curves will be used in many applications. That will be dealt with in chapter 2. This knowledge originates, like much of the rest of the geometry you have done this year, in ancient Greece. The modern terms parabola, ellipse and hyperbola are derived from the work of Apollonius from Perga (200 before Christ). This is also true for the word *asymptote*. Asymptotes play a role in hyperbolas.

Chapter 1
Edge and conflict


Greenpeace activists at the Brent Spar oil platform, 1995



THISTLE
MAGNUS
DUNLIN
EIDER
MURCHISON
TERN
STATFJORD
N. CORMORANT
GULLFAKS
S. CORMORANT
BRENT
HEATHER
3/4
CLAIR
TROLL
HUTTON
N. ALWYN
NINIAN
ALWYN
OSEBERG
Bergen
ODIN
N.E. FRIGG
FRIGG
E. FRIGG
Oslo
BRUCE
BERYL
HEIMDAL
CRAWFORD
BALDER
Stavanger
GUDRUN
BRAE
PIPER
TIFFANY
CLAYMORE
SLEIPNER
TONI
TARTAN
BEATRICE
THELMA
MAUREEN
BRISLING
BREAM
BUCHAN
FORTIES
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MONTROSE
Aberdeen
LOMOND
COD
ULA
5607
ALBUSKJELL
TOR
S.E. TOR
FULMAR
CLYDE
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35
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GORM
DAN
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E
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ROUGH
AMETHYST
F/18
WEST SOLE
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K/6
L/4
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L7
VIKING
K8, 11
L/12
L/15
BROKEN BANK
INDE
L/10
DOTTY
DEBORAH
HEWETT
K/14
L/14
GRONINGEN
K/13
L/13
K/17
K/18
HOORN
LEMAN
HELM
P/6
Q/8
HELDER
Q/7
Oil field
Gas field
Amsterdam
The Hague
NORTH SEA
0
100
200 Km
London
Hamburg

1: Borders under water

The border between the Netherlands and Germany does not end at the waters of the Dollard. This border progresses on the seabed.
On the map on the left side you can see that the Netherlands also have a long common border with Great Britain, and that we are very close to being direct neighbors with the Norwegians. You can also see why these borders are so important: the proprietary rights of the oil- and natural gas fields must be arranged appropiately.
These borders were determined in 1945. For example, on the Danish-Norwegian border lie all points equidistant from the shores of each of those countries.

1 **a.** A circle is sketched around two points on this border.
What meaning do these circles have?
b. ***If*** *P* lies on the Norwegian-Danish border, ***then*** d(*P*, *Norway*) = d(*P*, *Denmark*).
Is the reverse also true:
if d(*P*, *Norway*) = d(*P*, *Denmark*), ***then*** *P* on the Norwegian-Danish border?

distance point-regions

In chapter 5 of the book DISTANCES, EDGES & DOMAINS we identified the distance from a point to the area like this.

The distance from a point *P* to an area *A* is equal to the radius of the smallest circle round *P*, which has at least one point in common with the edge of *A*.

In this chapter distances will again play an important role. So we recall some earlier knowledge. You might have all this knowledge at hand. If not, consult chapter 5 of DISTANCES, EDGES & DOMAINS.

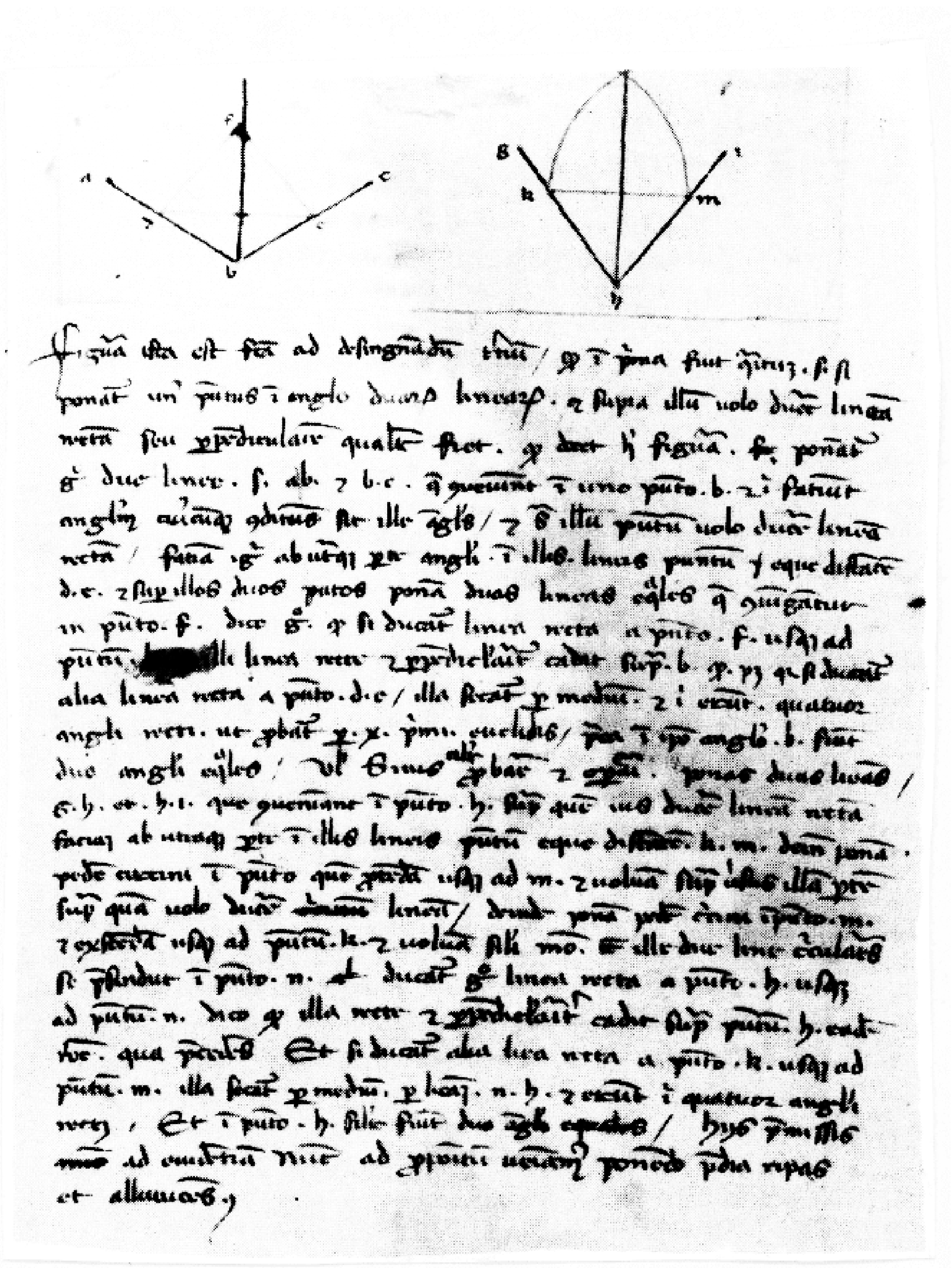
2 **a.** What do you understand by a foot (point)?
b. Can one point have different feet on the edge of a region?
c. What is an iso-distance line?
d. How can you sketch an iso-distance line for a complex region?

3 The Norwegian-Danish border ends in a three-countries-point.
a. Which country is the third country in question?
b. Show that this point truly has equal distances to the three countries by drawing the right circle.
c. South of this point lie two other three-countries-points. Indicate for each point which three countries are involved.

The circles of exercise **1a** can be called *largest empty circles* again, as was done when working with Voronoi-diagrams.
If you sketch circles around the two points meant in **3c**, which touch the concerned countries, you will see that there are no feet along the German shores. There is a historic reason for this. During the division in 1945 people were not concerned with the German interest. When later on, the international community did want to involve Germany in international conventions, the Netherlands and Denmark both had started their drillings. For the demarcation of the German sector some political compromises were made.

4 Sketch the original course of the Dutch-Danish under water border.

5 The borderline between the two regions is also called *conflict line*. A good name?

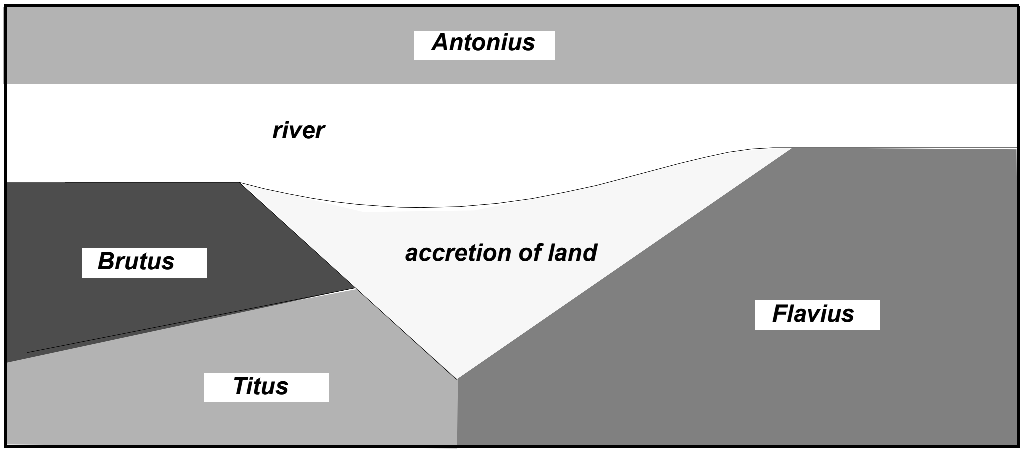


Here you see a page from Bartolus' book where the construction of the bisector of an angle is shown.

The original manuscript is in the Vatican Library.

2: Division problems

In this section we move to a river in the Roman Empire.
Once the river had a big bay on the south side. Over the years this bay gradually became silted up. This new land is very fertile and each of the neighbors wants to add as large a piece as possible to his domain.
But according to what rules must the alluvium by divided between the neighbors Brutus, Titus and Flavius? And can the opposite neighbor Antonius maybe also claim part of this land?

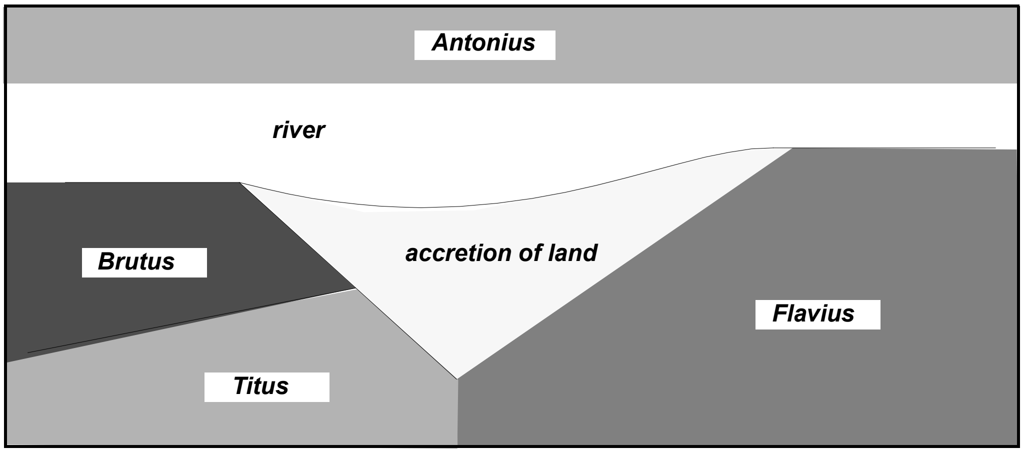


In Roman law, accretion of land was considered as *accessio*, increase: the increase to something which is my property is in itself also my property. According to this principle the owner of a cow also becomes the owner of the calf and the owner of a tree also becomes the owner of the fruits the tree bears. If you take the principle literally, the land must be divided among Brutus, Titus and Flavius, since the land has 'grown' onto their land. But how do the borders progress?

6 Brutus suggests extending the borders in the direction to run..
- **a.** Construct a possible division of the accretion of land according to this idea.
- **b.** Titus does not agree with this division. With which arguments could he make out a case?

The Roman jurist Gaius (second century after Christ) already went into this kind of problems in his *Institutiones*. But the Italian jurist Bartolus the Saxoferrato (1313 - 1357) was the first to realize that the problem of accretion of land was basically a mathematical one. Inspired by the conflict which he experienced during a vacation by the Tiber, he wrote a treatise under the title *Tractatus the fluminibus* (*flume*n is the Latin word for river) dealing with the problem of accretion of land. In this work he determined the 'conflict lines' with geometrical instruments for a large number of situations. He used the same criterion as we will in this chapter, namely: *the smallest distance to the old shores determines who will be the owner.*

7 Divide the accretion of land among the neighbors Brutus, Titus and Flavius according to that principle.



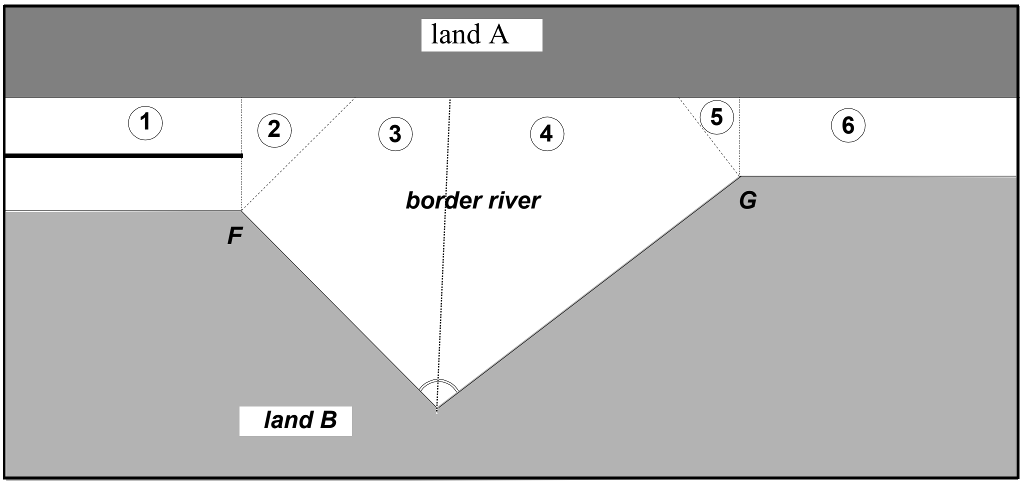
Antonius (other side of the river!) does not want to accept that he has no role in the division. He secured an expert's support and challenged the decision in a court of law. According to the expert, Antonius is also entitled to a small part of the alluvium. Since it is hard for him to cultivate that part, he is willing to sell it to one of his opposite neighbors.

8 Find out whether Antonius does have the right to part of the alluvium.

bisector

In the solution of division problems bisectors (whether or whether not of a straight angle) play an important role.
This is also true for the following problem.

9 The border between the countries *A* and *B* must lie exactly in the 'middle' of the border river. To find this border the river is divided into six sectors trough four perpendicular lines and one bisector (in the figure those five lines are dotted).



midparallel

a. In sector 1 the borderline between *A* and *B* has been drawn. The line there is the midparallel of the two parallel banks.
For which other sector is the borderline also a midparallel?

b. In two sectors, the border lies on a bisector.
Sketch these parts of the borderline.

c. In the two remaining sectors the borderline is curved. Try to sketch those borders as accurately as possible.
d. Why are these curved lines *not* circle arcs?
e. Which role do the points *F* and *G* play for the 'edge' of *B* ?
(Think about problems round iso-distance lines)

conflict line, conflict point

All three problems revolved round a set of *P* with the property:

d(*P* , region *X*) = d(*P* , region *Y*).

Henceforward we will call such a point a *conflict point*.
The set of all conflict points is called *conflict line*.
Thus:

> **The conflict line of two areas A_1 and A_2 is the set van all points *P* for which hold: d(*P* , A_1) = d(*P* , A_2).**

Examples:
– if the edges of the areas are two parallel lines, then the conflict points lie on the mid-parallel of these lines;
– if the edges of the areas are non-parallel lines, then the conflict points lie on the bisector of the angle, which (after extension of the edges is necessary) is enclosed by these lines.

If the edges of the areas have a whimsical shape, then the conflict line can still be fairly straight as was seen in the Norwegian-Danish border on the bottom of the North Sea.
A conflict line can also be curved; that was the case in exercise **9d**.
The demand for the conflict points there was as follows:

d(point , point) = d(point , straight line).

This last case and other similar cases will be looked into in the next sections.

3: Construction of conflict points

In this section we start with a systematic research for the conflict lines between two simple areas. This research will be continued in section 4 with a computer practical and will be finished in following chapter.
We investigate all cases in this table.

conflict line between ... and ...	point	straight line	circle
point	perpendicular bisector		
straight line		pair of angle bisectors, midparallel	
circle			

Two cells have been filled. Remark: in the table the (straight) lines and circles are not meant as shores of regions, but they are (abstract) areas themselves. That is why you find a *pair* of bisectors in the middle cell. Also a point is considered to be an area. The well known theorem of the perpendicular bisector between two points allies beautifully with the definition of a conflict line.

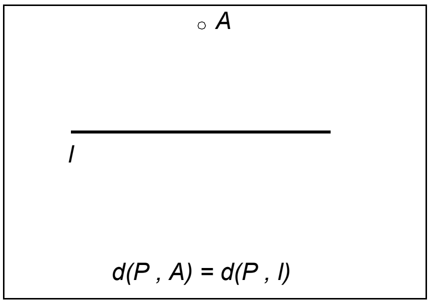
perpendicular bisector

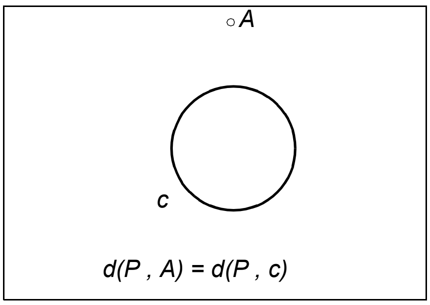
The perpendicular bisector van two points *A* and *B* is the set of all points *P* for which hold: $d(P, A) = d(P, B)$.

The midparallel and the bisector came up in the previous section.

10 Explain why the set of all conflict points of two intersecting lines is a bisector pair.

Due to the symmetry four cases remain in the table.
Below you see two sketches for the cases ‘point - straight line’ and ‘point - circle’.



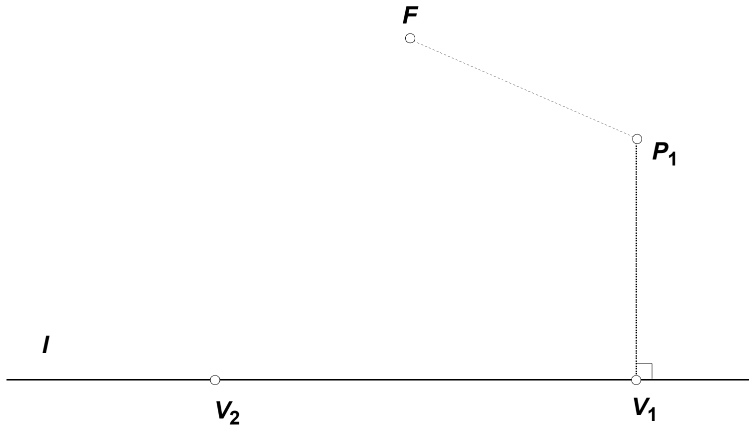


11 a. Try to sketch the conflict line in the lefthand figure.
First consider:
- of which conflict points can you indicate the exact location easily;
- how you find the foot of an arbitrary conflict point on line *l*;
- which symmetry the conflict line has;
- whether the conflict line is infinitely long or closed.

b. Sketch, after the same considerations, the conflict line in the righthand figure.

In a CABRI-practical we want to study these cases extensively. But to get these conflict lines on the screen we have to come up with a construction method. We will first scrutinize the case point - line.

the case point-line



12 In the sketch you see at the point F and the line l a conflict point P_1 and its foot V_1.
a. How can you verify that $d(P_1, F) = d(P_1, V_1)$?
b. Is P_1 the only conflict point that has V_1 as a foot? Explain why/why not.

13 In the sketch another point V_2 is indicated on line l. Starting with V_2 we execute the construction again, but now we incidentally use some appropriate terminology
a. Sketch all points of which V_2 is the foot point on line l. What is this collection of points?
b. Draw all points equidistant from V_2 and F. What is this set of points?
c. Now mark the wanted point P_2: the conflict point of F and l with foot V_2 on l.

14 Using this method you are able to construct a conflict point P for *each* point V on l.
a. Why does the intersecting of the two lines (13a and 13b) never go wrong?
b. Summarize the method in a construction scheme, which we will execute later with CABRI. Complete this scheme.

How do you construct a point on the conflict line between a point and a line?

1. Draw the point *F*.
2. Draw the line *l*.
3. Draw a point *V* on *l*.

preparation phase

4. Draw ....
5. Draw ....
6. Mark the intersection and call this point *P*.

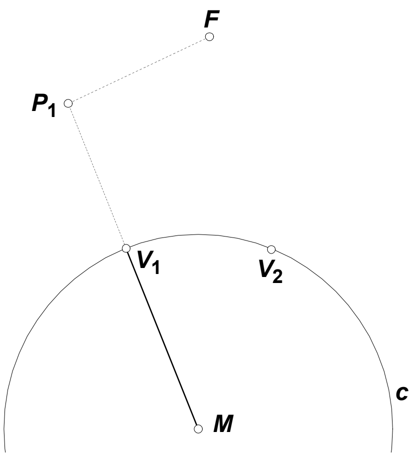
construction phase

With this scheme you can construct any conflict point. You show this construction to CABRI once very precisely and after that you let CABRI do the construction for all the other points on the line l, using the Locus-option

the case point-circle

15 Now the case point-circle.
The figure shown below looks a lot like the figure in the case point-line. Point P_1 lies on the conflict line since $d(P_1, F) = d(P_1, V_1)$.
a. V_2 is the foot of another different conflict point P_2.

Demonstrate that construction of P_2 can be executed in practically the same way as was done for the case point-line.



How do you construct a point on the conflict line between a point and a circle?

1. Draw the point *F*.
2. Draw the circle *c*.
3. Draw a point *V* on *c*.

preparation phase

4. Draw ....
5. Draw ....
6. Mark the intersection and call this point *P*.

construction phase

b. Complete the schedule shown above.
c. Do you find a conflict point *P* for *each* point *V* on the circle this way?
d. Do you find *all* points of the conflict line this way?

In applications it goes without saying to interpret the circle *c* as the edge of a region. In exercise **15** you are so to speak looking for the conflict line between a large island and a tiny, but independent island, which lies a few miles offshore.
However, point *F* can also lie in the interior of circle *c*. In that case you could think of a land with an inland lake, wherein a 'very small' independent island is located.

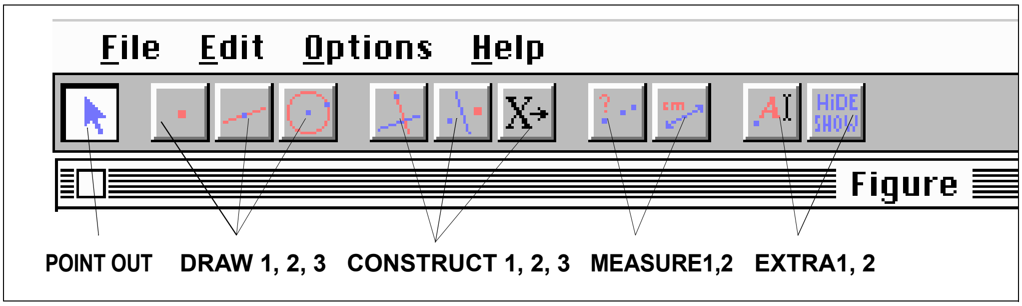
16 a. Draw a circle with a point *F* in the *interior* area of the circle and sketch the conflict line between *F* and the circle..
b. Check whether or not you construct all points of this conflict line with the construction scheme of exercise **15b**. If not, make a appropriate scheme for it.
c. Suppose *F* happens to be the center of circle *c*. What is the conflict line in this case?

circle-line and circle-circle come later on

At this moment and in the next CABRI-practical we do not yet pay attention to the two cases 'circle-straight line' and 'circle-circle'.
Later on it will become clear that these cases can be reduced fairly easily to the familiar cases 'point-straight line' and 'point-circle'.

4: Conflict Lines with CABRI

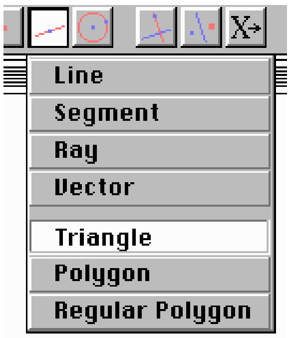
Below, you see again the tool bar of CABRI. We use the same names for the buttons as we did in the book THINKING IN CIRCLES AND LINES. There we used CABRI during the orientation for several proofs (thus as a tool when looking for proofs), now we use CABRI for the construction of conflict lines.



operation

In this practical you use the options from the tool bars, which are under the buttons DRAW 1,2,3, CONSTRUCT 1 and EXTRA 1,2.
We will indicate options again like BUTTON > OPTION. The task 'draw a triangle' will look like this:

DRAW 2 > TRIANGLE



Then you click on button DRAW 2 and drag the pointer to the option TRIANGLE.

Since you might have forgotten how to work with CABRI you will find an overview of all *options* in the back of this book (see page 47).

The goal of this practical is:
- to construct the conflict lines for all of the cases point-line and point-circle;
- to investigate for both cases all different situations;
- to interpret the results and translate those into definitions for parabola, hyperbola and ellipse.

configurations

17 Before you start working on the construction exercises, you might need to adjust three configurations of the program.
Open the bar OPTIONS > PREFERENCES.
- Set NUMBER OF OBJECTS IN LOCUS on **500** (or more; but maybe some drawings will overfill).
- Set LOCUS OF POINTS: LINK POINTS on **ON** (you see a cross in the square).
- Set LOCUS OF LINES: ENVELOPE on **OFF** (thus no cross in the square).

Finally click on APPLY TO to activate the settings.

the case point-line

18 We first investigate the case ‘point-straight line’.
We found the steps of the preparation phase and the construction phase in the previous section. The last three steps are to clarify the drawing as much as is possible

How do you construct a point on the conflict line between a point and a line?

1. Draw a point *F*.
2. Draw a line *l*.
3. Draw a ***new*** point *V* on *l*.

(preparation)

4. Draw the perpendicular line in *V* on *l*.
5. Draw the perpendicular bisector of *V* and *F*.
6. Mark the intersection of the two lines.

(construction)

7. Put names at the points *F*, *V* and *P*.
8. Draw the line segments *FP* and *VP* and color them green.
9. Draw line segment *FV* and color this line segment yellow.

(embellishment)

a. Execute this construction.
You can mark an intersection point with the option DRAW 1 > INTERSECTION POINT.
You can label points with the option EXTRA 1 > LABEL.
You can color objects with the option EXTRA 2 > COLOUR.

b. Now drag away point *V* and look what happens with point *P*. This point now walks on the conflict line. Watch triangle *FPV* carefully. What does and what does not change?

c. With the option EXTRA 1 > ANIMATION you can use the computer so to speak as a pinball machine. Choose this option and click on point *V*. When you now try to drag point *V* away, you see a string being stretched. Try it and let go of *V*.
In addition to this:
- The further you move the mouse, the more the spring is tightened and the faster point *V* will move along line *l*. So do not pull too hard!
- It could be that in the end *V* comes to a standstill outside your screen; with the option EDIT > UNDO you return to the situation before the ‘launch’.
- You can stop the animation at any given moment by clicking somewhere in the drawing or with ESC.

d. Up to now you have seen point *P* moving, but you have not seen the conflict line itself.
You make this curve visible with the option CONSTRUCT 1 > LOCUS.
Choose this option. **First** click **on *P* and then on *V*.** The computer now sketches the conflict line.

locus = place

Remark: ‘Locus’ is the Latin word for ‘place’. With the option LOCUS you sketch the place of all points *P* when *V* moves along line *l*.You may also speak more modernly of the orbit of *P*, when moved by *V*.

19 Now investigate how the shape of this conflict line changes when you enlarge or decrease the distance from *F* to line *l*. You can also turn *l* and see what happens.

20 When you drag point *V*, the perpendicular bisector of *FV* turns.
- **a.** Which role does this perpendicular bisector appear to play for the conflict line?
- **b.** With the LOCUS-option you are also able to sketch many positions of this perpendicular bisector in one go. Think which actions you need to take and let the computer sketch this locus.

parabola

The conflict line between a point *F* and a straight line *l* is called a *parabola*.
In the next chapter we will give an exact geometrical definition of the parabola.

21 Write down in the survey what you already know of parabolas.

> A parabola is the conflict line between ....
> A parabola looks symmetrical in ....
> The vertex of the parabola is the middle of ....
> The parabola looks narrower/wider as the distance between focus and directrix decreases.
> The .... appear to be tangent lines to the parabola

the case point-circle, part 1

We now investigate the case point-circle.

We first investigate the situation where point *F* lies outside the circle.

22
- **a.** Clean the drawing screen.
- **b.** Adjust two lines in the construction scheme for the parabola, now you already have the construction scheme for the new conflict line.
  Use in line 4 the option DRAW 2 > RAY to draw a 'half line', which begins in point *M*. First click on *M* and then on *V*, or the ray will go in the wrong direction.
- **c.** Now execute the construction.
- **d.** Do the same investigation steps as in exercise **18**, thus
  - first drag *V* by hand
  - use the 'pinball machine'-option (ANIMATION)
  - draw the conflict line with the LOCUS-option
  - draw the perpendicular bisectors *pbs*(*F*, *V*) with the LOCUS-option.

For the parabola it looked like the perpendicular bisectors filled the whole exterior of the parabola. In this case it looks like the perpendicular bisectors leave a whole area free, which resembles the interior region of the conflict line!

23
- **a.** Now look again what happens when you drag *V*. The perpendicular bisector and the half line do not always intersect.
  Now adjust line 4 in the construction scheme in such a way that the computer does not sketch the *half* line, but the *whole* line through *M* and *V*.
- **b.** Remove the half line from the screen and execute the new construction. If you now sketch the set of points *P* with the LOCUS-option, you notice that the sketched figure consists of two parts, i.e. has two *branches*. These two branches are each separately mirror symmetrical, but they also look like each other's images. Which two lines are the symmetrical axes?
- **c.** There are two points *V* on the circle for which the construction does not give a

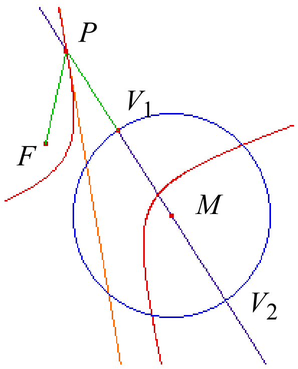
point P on either of the branches. Try to find these two places on the circle.

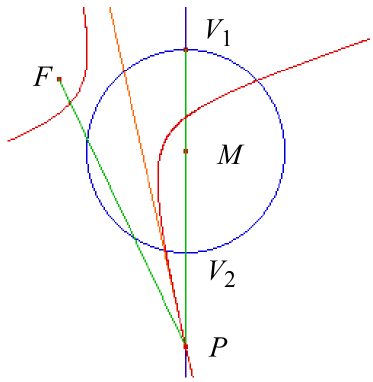
d. Which shape does the triangle FVM have in these cases?

e. Which role does the perpendicular bisector $pbs(F, V)$ play for the conflict line in these cases?

f. And which role does the perpendicular bisector $pbs(F, V)$ seem to play if V lies elsewhere on the circle?

24 Here you see two screen prints. In the lefthand figure you see P a point of the branch, which lies completely outside the circle. In the righthand figure, P lies on the branch which intersects the circle. The line through P and M intersects the circle in points





V_1 and V_2 in both figures.
Answer these questions for each of the figures.

a. Which of the points V_1 and V_2 is *according to the figure* the foot of P on the circle?

b. And which of the two points is *according to the construction* the foot of P?

c. To which of these points applies $d(F, P) = d(V, P)$?

d. What can you say about these two distances for the other point?

hyperbola The conflict line between a circle and a point outside the circle is one of the two branches of a *hyperbola*. In the next section we will give an exact definition of the hyperbola.

25 Insert your results in the following short summary.

A hyperbola has two branches.
The conflict line between a circle and a point F outside the circle is ....
....
A hyperbola looks symmetrical in ....
The hyperbola looks narrower/wider as the distance between point F and the circle decreases.
The .... appear to be tangent lines to the hyperbola.
For two points V on the circle the construction does not give a point P of the hyperbola. In these cases
– is triangle VFM ....
– is $pbs(V, F)$ .... of the hyperbola.

the case point-circle, part 2

26 Now drag point F to the interior region of the circle. If necessary, first increase the circle using the pointer and pull.
Also execute all investigation steps for this situation, thus:
- first drag V by hand, try to understand what happens
- then use the ANIMATION-option
- then sketch the conflict line with the LOCUS-option; change the place of A within the circle and investigate the consequences for the shape of the conflict line
- sketch the perpendicular bisectors $pbs(A, V)$ with the LOCUS-option.

ellipse

The conflict line between a circle and a point A within the circle is an *ellipse*.
In the next section we will give an exact definition of the ellipse.

27 **a.** An ellipse appears to have two symmetrical axes. Which two lines are these?
b. Draw these two lines with CABRI. Change the shape of the ellipse and look whether these two lines remain the symmetrical axes of the ellipse.
c. Now drag point F in such a way that it coincides with the center of the circle. What happens to the ellipse?
And what can you say about symmetrical axes in this case?

28 Now make a short summary of the most important results for the ellipse.

An ellipse is the conflict line between..
An ellipse looks symmetrical in..
The ellipse looks narrower/wider as the distance between point F and
the circle decreases.
The .... appear to be tangent lines to the ellipse.
....

Chapter 2
Parabola, ellipse and hyperbola

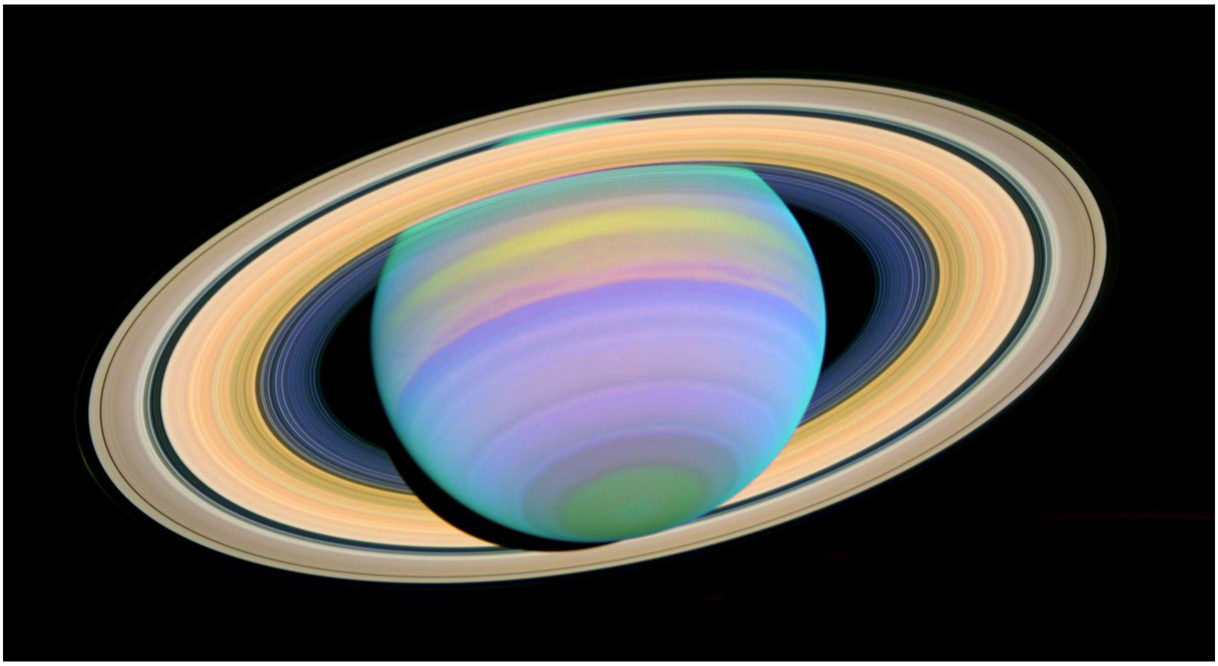


Photo other side: Saturn, photographed with the Hubble Space Telescope.

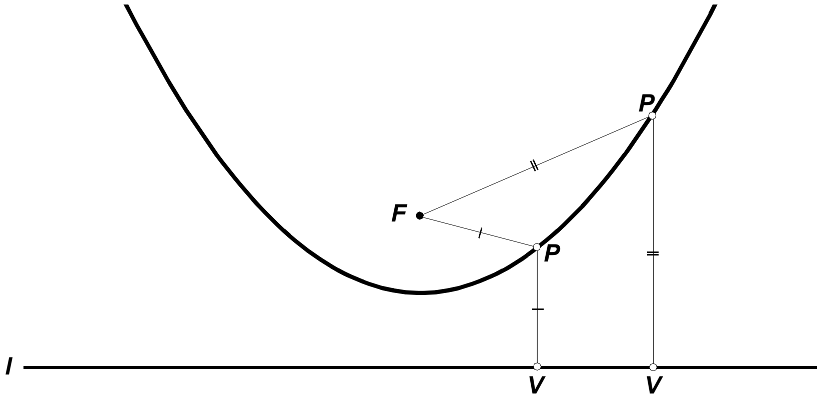
The planet Saturn revolves around the sun in 29 years at a distance of 1 429 400 000 km and has a diameter of 121 000 km.
Round its equator, a gigantic storm rages, big enough to make the whole earth disappear. See the white spot on the photo.
Saturnus has a number of rings, which can be seen clearly on the photo. These rings are circles, but since you are looking at them from an angle, they are distorted.
The Hubble Space Telescope rotates around the earth in a slight elliptic orbit at a height of approximately 600 km. Due to the good atmospheric circumstances at that height – no atmosphere, to be precise – it is possible to take very detailed photos.
This telescope does not use lenses, but a curved mirror. The main mirror of the system is a parabolic mirror with a diameter of 2.5 meter.

5: The way the parabolic mirror works is explained in the geometry of this chapter. Definitions and properties of parabola, ellipse and hyperbola

what is a parabola?

definition parabola

Let *F* be a point and *l* a straight line, which does not go through *F*.
The set of all points *P* for which hold: d*(P , F)* = d*(P , l)* is called a *parabola.*
***F* is called the *focus* of the parabola, *l* is called the *directrix* of the parabola.**



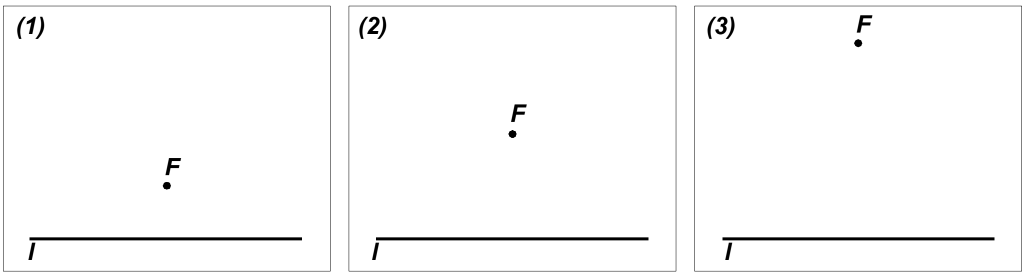
That *F* is called a focus originates from a beautiful application: the parabolic mirror. This will be explained in section 7 of this chapter.

1 Is the definition much different from the description given earlier on?

symmetry, vertex

A parabola is thus completely determined by a point *F* and a straight line *l* (not through *F*).
The figure formed by *F* and *l* is symmetrical, and therefore so is the parabola.
The point of the parabola, which lies on the axis of symmetry, is called the *vertex* of the parabola. It is the point of the parabola which has the smallest distance to the directrix and focus.

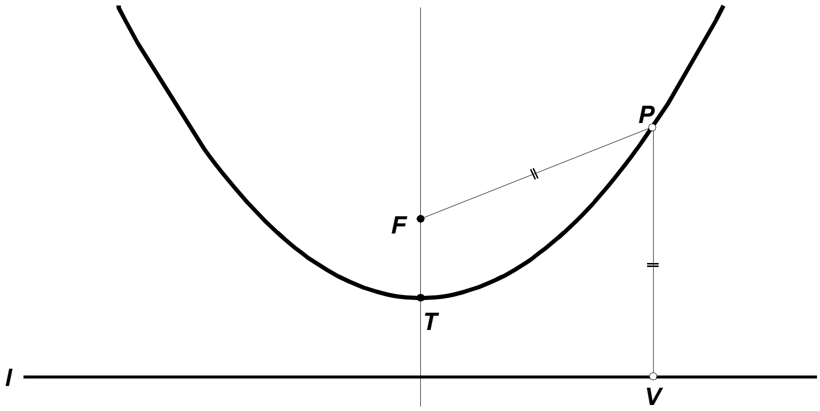
2 If you have the vertex and a couple of points (like the ones exercise **11 a**, page 3), you can already make a quite reasonable sketch of the parabola. Do this in the three situations shown below.



all parabolas are similar

It looks like you can make a distinction between 'narrow' and 'wide' parabolas, but that depends! When you enlarge the parabola of (1) so that the distance of *F* to *l* is equal to the distance in (3), then you get the parabola of (3). By zooming in or out you can make any two parabolas equal to each other. Therefore we say that all parabolas *are similar.*

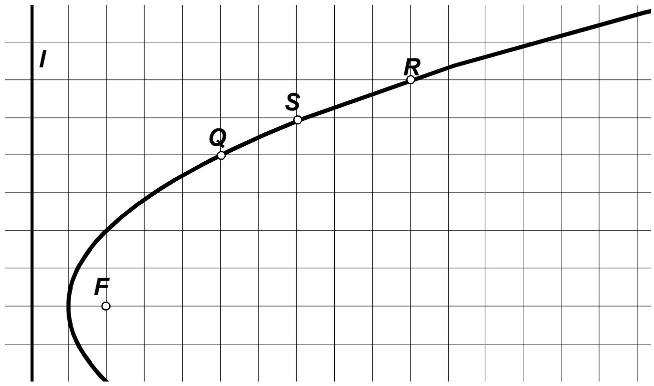
3 Here the axis of symmetry is indicated in the figure.



a. Check whether the figure is drawn correctly by sketching a circle with center *P*, which is tangent to *l*. What should be correct now?

b. Show that there is another parabola with the same directrix and axis of symmetry, which also goes trough *P*; determine the focus of this parabola very precisely.

4 Here a parabola is sketched with a square grid in the background. Focus and directrix lie exactly on this grid.

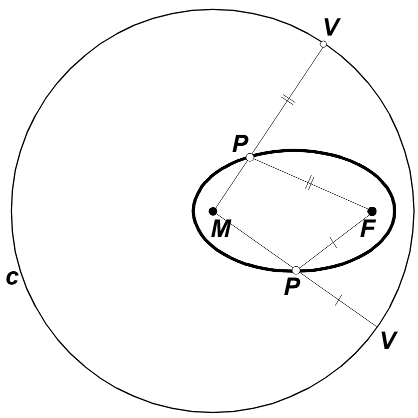


Show trough computation that the grid points *Q* and *R* lie on the parabola and *S* does not according to the definition of the parabola.

what is an ellipse?

We described the ellipse as the conflict line between a circle *c* and a point *F* within the circle. You can immediately – just as for the parabola – make a definition from it. It would probably look like this:

> Let *c* be a circle and *F* a point in the interior region of *c*.
> The set of all points *P* for which hold: d(*P*, *F*) = d(*P*, *c*) is an ellipse.



But this is not very pretty due to the following. An ellipse is completely determined by a circle *c* and a point *F* (within *c*). The figure formed by *F*

and c has one axis of symmetry (the line which connects F with the center M of c). This line is therefore also a axis of symmetry of the ellipse.
The ellipse turned out to have a second axis of symmetry, namely the perpendicular bisector of MF. This cannot be seen directly from looking at the temporary definition, but will become clear in the next exercise.

5 Look at the figure on page 22. The circle c has radius r.
- **a.** Explain that the condition $d(P, F) = d(P, c)$
  is equivalent to: $d(P, M) + d(P, F) = r$.
- **b.** Explain why the perpendicular bisector of MF is a axis of symmetry of the ellipse.

In the second condition of exercise **5a** the points F and M do no longer play different roles; this was the case originally. In order to emphasize the equivalence, we will call these points F_1 and F_2 from now on. So we have finally:

standard-definition of the ellipse

> **F_1 and F_2 are two different points.**
> **The set of all points P for which the *sum* of the distances to F_1 and F_2 is constant, is called an *ellipse*.**
> **F_1 and F_2 are called the *foci* of the ellipse.**

Remark: the constant must be bigger than the distance between the foci.
We will return to the term *foci* later.

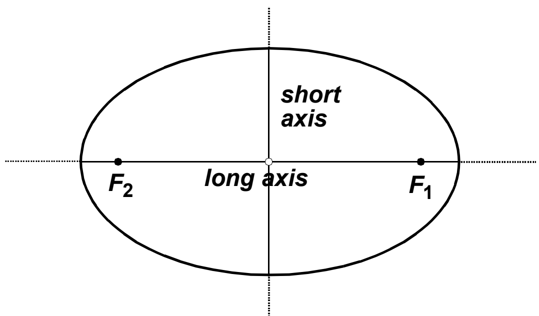
6 Here is a way to create an elliptical flowerbed in the garden.

> **Put two poles in the ground at two meters distance from each other.**
> **Knot the ends of a 9 meter long rope together so you get a loop.**
> **Put the loop around the poles and pull the rope taut with a stick.**
> **Move the stick around and scratch an oval in the ground with the point.**

- **a.** Why does a flowerbed made in this way have the form of an ellipse?
- **b.** How long and how wide will this ellipse be?
- **c.** How do the length and the width of the ellipse change if you put the two poles at a smaller distance than 2 meter from each other in the ground?
- **d.** It is also possible to make a three meter wide ellipse with the rope.
  At what distance from each other should you put the poles in the ground?

long and short axis, vertices

You have seen that an ellipse has two symmetrical axes:
– the line that connects the foci;
– the perpendicular bisector of the two foci.



The segments of the symmetrical axes which lie inside the ellipse are called:
long axis and *short axis*. Both foci lie on the long axis.

The four intersections of the axes with the ellipse are called the *vertices* of the ellipse.

7 Of a point P it is given that it lies on an ellipse with foci F_1 and F_2 and that $d(P, F_1) = 7$ and $d(P, F_2) = 3$ hold. Additionally $d(F_1, F_2) = 8$ holds.
How long are both axes (long axis and short axis) of the ellipse?

8 **a.** Two different ellipses do not have to be similar (like two parabolas). That is, (most of the time) it is not possible by zooming in or out to make two ellipses equal to each other. This follows for example from the temporary definition where an ellipse is determined by a circle c and a point F. Explain this.
b. What kind of shape does the ellipse have when F coincides with the center of c? And what do you then know about the long axis and the short axis?
c. Suppose you have two similar ellipses. Which property should axes of these ellipses have?

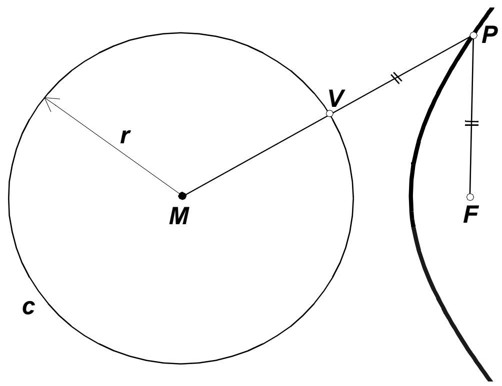
9 Given two points F_1 and F_2 and a number $r > d(F_1, F_2)$.
c_1 is the circle with center F_1 and radius r; c_2 is the circle with center F_2 and radius r.
Show that the conflict line of F_1 and c_2 coincides with the conflict line of F_2 and c_1.

director circle

In the standard definition of the ellipse the circle c does no longer play a role. When constructing an ellipse and in several reasonings, it will come in very handy. That is why this circle has a special name: it is a *director circle* of the ellipse. It is a circle with radius, the constant from the definition, and center, one of the foci. Thus there are two circles directrices.

what is a hyperbola?

The conflict line of a circle and a point outside that circle was just one branch of a hyperbola. This is why we let the standard definition of the hyperbola differ even more from the conflict line description than was done for the ellipse .
In the figure below, you see one hyperbola branch, sketch according to the conflict line description.



Such a hyperbola branch is completely determined by a circle c and a point F outside c. The figure formed by F and c has one axis of symmetry (the line, which connects F with the center M of c). This line is therefore also a axis of symmetry of the hyperbola branch. The point of the hyperbola branch, which lies on the axis of symmetry, is the vertex of the hyperbola branch.

10 Explain that the condition: $\mathrm{d}(P, F) = \mathrm{d}(P, c)$
is equivalent to: $\mathrm{d}(P, M) - \mathrm{d}(P, F) = r$.

With this last condition we are close to the definition of a hyperbola.
But the points M and F here are not, as for the ellipse, interchangeable. For the sum of two distances the order is not important, for a difference it is.

11 a. Verify that the set of points P which satisfy $\mathrm{d}(P, F) - \mathrm{d}(P, M) = r$ are in fact a new hyperbola branch, namely the mirror image of the hyperbola branch which belongs to the condition $\mathrm{d}(P, M) - \mathrm{d}(P, F) = r$.
b. For which circle and which point is the new hyperbola branch the conflict line?

absolute difference

The hyperbola branch meant in **11 a** forms a complete hyperbola together with the other branch. Or: the point P lies on the hyperbola when:

$$\mathrm{d}(P, M) - \mathrm{d}(P, F) = r \quad \text{ór} \quad \mathrm{d}(P, F) - \mathrm{d}(P, M) = r.$$

Shortly noted: P lies on the hyperbola when:

$$\mathrm{d}(P, M) - \mathrm{d}(P, F) = \pm r$$

Or, using the *absolute value:*

$$|\ \mathrm{d}(P, M) - \mathrm{d}(P, F)\ | = r$$

Say:

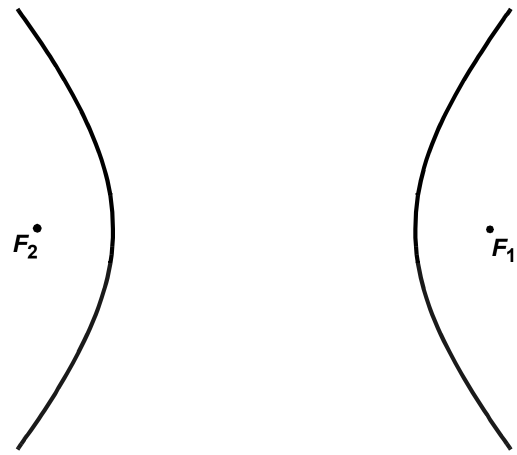
the absolute difference of $\mathrm{d}(P, M)$ and $\mathrm{d}(P, F)$ is equal to r.

In order to emphasize the fact that the points M and F are equivalent in this condition, we call them F_1 and F_2. The definition of the hyperbola, as can be found in most books, finally comes out as:

standard definition of the hyperbola

> **F_1 and F_2 are two different points.**
> **The set of all points *P* for which holds:**
> $$| d(F_1, P) - d(F_2, P) | = r$$
> **is called a *hyperbola.***
> **F_1 and F_2 are called the *foci* of the hyperbola.**

12 Here a hyperbola with foci F_1 and F_2 is sketched.



a. Determine what the used value of r is by measuring.
b. Choose a point P on the branch at F_1 and a point Q on the other branch, in such a way that the quadrangle F_1PF_2Q is a parallelogram. Which role does the middle of PQ play?
c. Now such that F_1PF_2Q is a rectangle. Using your protractor this should not be difficult.
d. Can you also make F_1PF_2Q square?

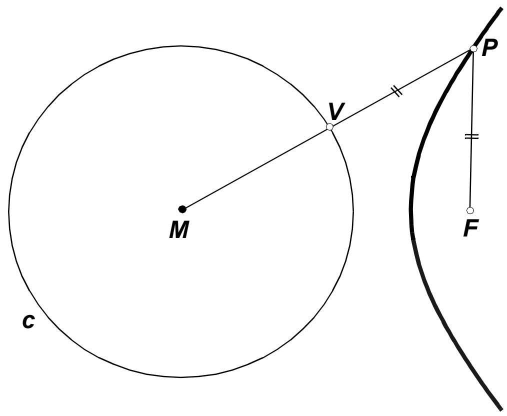
13 When speaking about a hyperbola it is somewhat off to talk about long and short axes, but it is reasonable beyond a doubt to talk about axes.
Why?

14 Describe what could be meant with the concept *director circle* of a *hyperbola*.

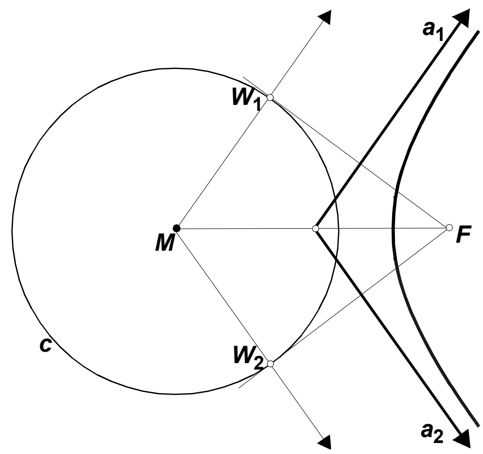
15 All parabolas are similar. Not all ellipses are similar.
What about hyperbolas?

the asymptotes of the hyperbola

A specific detail, which you discovered earlier for the hyperbola branch as conflict line, is that the feet of the conflictpoints P only lie on a part of the circle. Just look at the figure which belongs to the conflict description.

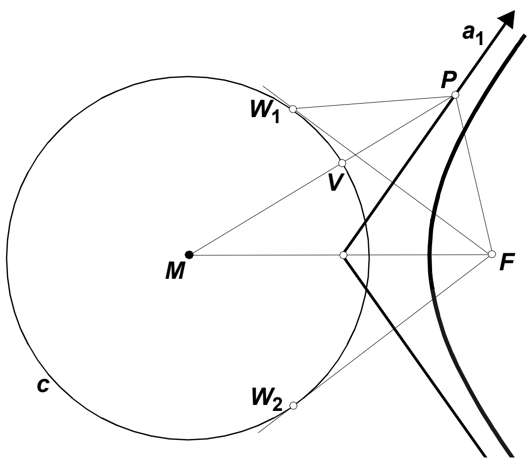


In the figure below, W_1 and W_2 are just the edge points where V must stay between. W_1 and W_2 are the tangent points of the tangent lines from F to the circle. The line MW_1 is parallel to the perpendicular bisector of FW_1 and thus does not yield a P. The same can be said about MW_2 and the perpendicular bisector of FW_2.



The perpendicular bisectors of FW_1 and FW_2 are called the *asymptotes* (say a_1 and a_2) of the hyperbola branch. When you move the foot V over the arc W_1W_2, then point P will move along the hyperbola branch. When V comes close to W_1 of W_2, then P moves very far away; then the distance to a_1 (of a_2) becomes very small. This distance approaches 0 as V approaches W_1 (of W_2). In an extra exercise (number **17**) this will be looked into.

16 It looks as if the hyperbola branch is completely formed *within* the angle made by the half lines a_1 and a_2. But you are not able to see what happens when V lies very close to one of the edge points.



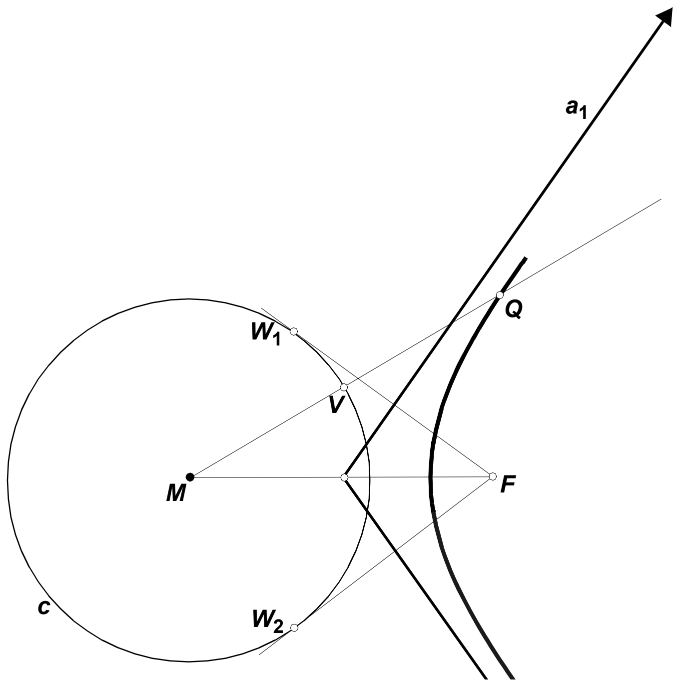
a. In order to prove that the hyperbola branch does stay within the angle, you take an arbitrary point P on one of the asymptotes. For that, it can be proven that it belongs to the sphere of influence of c, in other words that:
$d(P, V) < d(P,F)$
Prove the latter.

b. Point S lies on the same side from a_1 as W_1. Show that S certainly does not lie on the indicated branch of the hyperbola.

For a complete hyperbola – with both branches – the asymptotes are whole lines, it should be obvious.

extra, investigation asymptote

17 That the hyperbola comes 'infinitely close' to a_1 is somewhat more difficult to see. Here is an idea for a proof



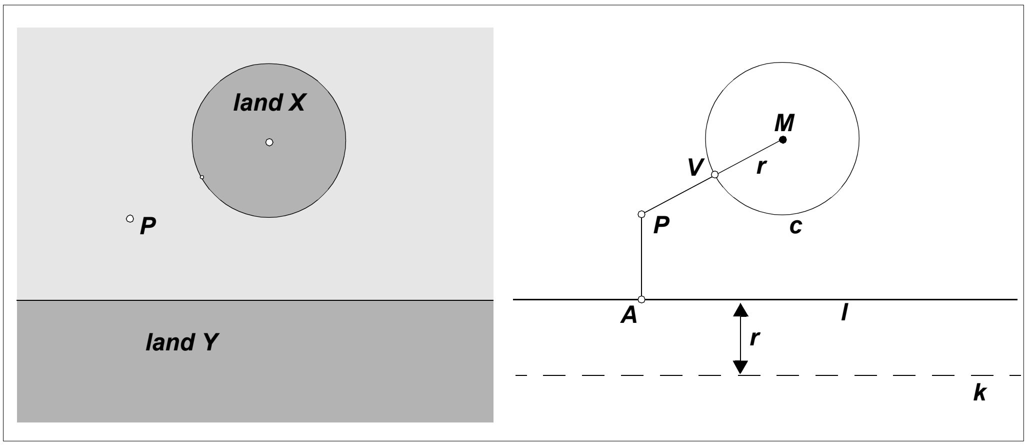
a. Add the perpendicular bisector of VF. It intersects W_1F in K.
b. What do you know about $\angle QKW_1$?
c. Q looks closer to a_1 than K. Why?
d. What happens with K when V approaches W?
e. Now finish the proof that Q can be as close to a_1 as you want..

6: Reduce and split Conflict Lines

The table in section 3, page 10, has now been filled out for the most part. You learn in this section how you can reduce the two missing cases to familiar situations.

conflict line between ... and ...	point	straight line	circle
point	perpendicular bisector	parabola	ellipse, hyperbola branch
straight line		pair of bisectors, midparallel	
circle			

the case line-circle



18 **a.** In the left figure a circular island lies opposite the straight shore of a neighbor country. Sketch the conflict line in this figure.

b. In the right figure, there is besides line l and circle c also a line k, which runs parallel to l at distance r.
For each point P on the conflict line holds $d(P, c) = d(P, l)$ thus $d(P, V) = d(P, A)$.
Explain that $d(P, M) = d(P, k)$ also applies to P.

c. Thus, which shape does the wanted conflict line have?

Remark:
In the last exercise we applied an important technique.
We reduced the circle, so to speak, to its center. After all, we were no longer looking for the conflict line between a *circle* and a ... , but for a conflict line between a *point* and a ....
As a result of this we had to
replace the distance $d(P, c)$ by $d(P, M) - r$.
The equation $d(P, c) = d(P, l)$ became $d(P, M) = d(P, l) + r$.

We could think of $d(P, l) + r$ as the distance from P to a line k, which runs parallel to l at distance r.

Thus: the condition $d(P, c) = d(P, l)$ is equivalent to $d(P, M) = d(P,k)$.

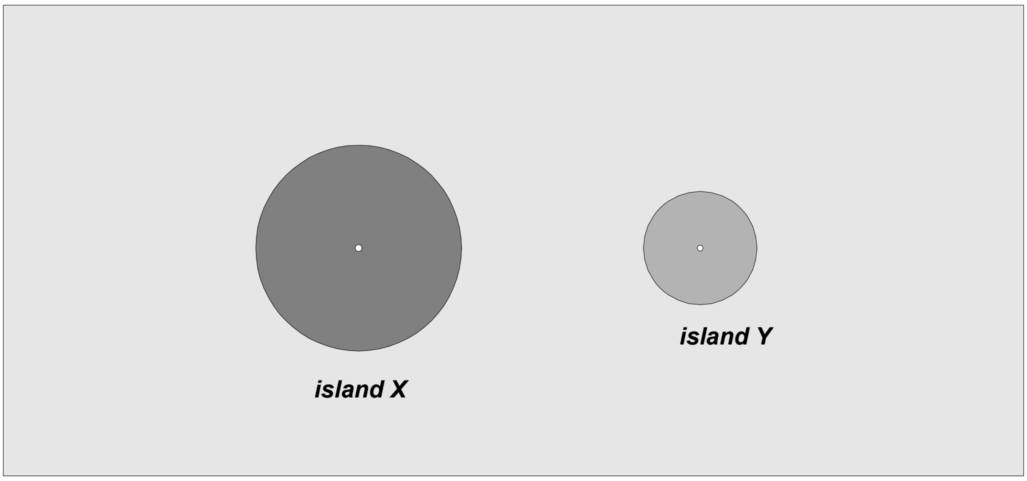
In this way we reduced the problem to a familiar situation.
We will apply this technique also to exercises **20** and **21**.

case circle-circle

In the case circle - circle we limit ourselves to the situations where the circles do not have common points. We first look at two special cases.

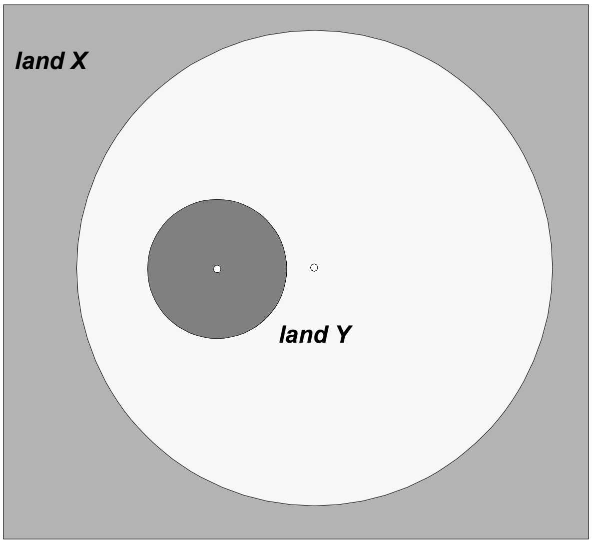
19 **a.** Which shape does the conflict line between two circles of the same size, which lie outside each other, have?
b. Which shape does the conflict line between two circles with different radius, but with the same center, have?

20 Now determine the conflict line between these two circular islands.



A good approach is:
– first draw a couple of points of the conflict line
– formulate an assumption: which familiar conflict line could it be
– make a new sketch with relevant data
– reduce the problem to a familiar situation.

21 The circular island *Y* lies in a circular inland lake of land *X*.
a. Argue which shape the conflict line of these two lands has.
b. Draw several points of the conflict line and then sketch the conflict line.

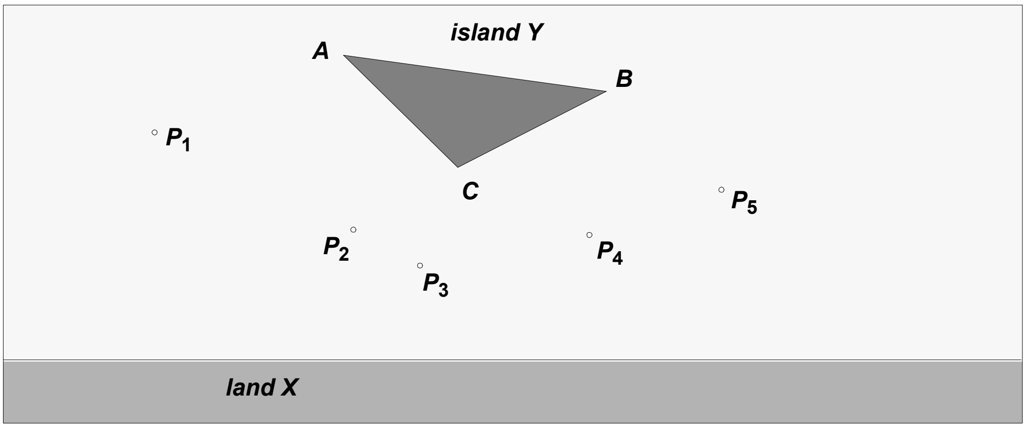


22 How does the shape of this conflict line change if
a. the radius of island *Y* increases/decreases?
b. the distance between the centers of the two circles increases/decreases?

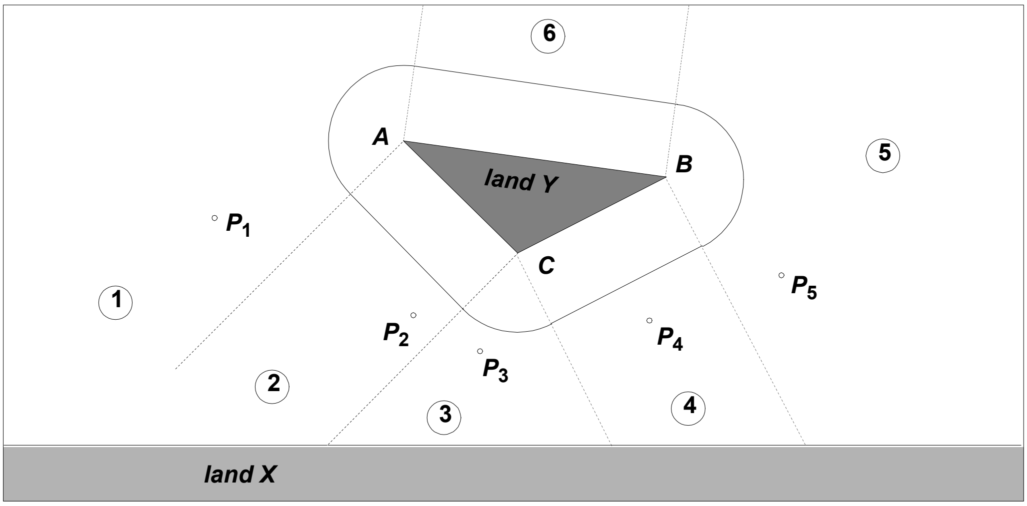
23 Process the results of exercises **20** and **21** into the survey scheme.

We now investigate a few situations where we need to split up the to be divided region in sectors. You also encountered such problems in chapter 5 of DISTANCES, EDGES AND REGIONS for iso-distance lines. Then capes played an important role.

24 A triangular island *Y* lies opposite the straight-lined shores of land *X*. In the region between the two lands five points are indicated.

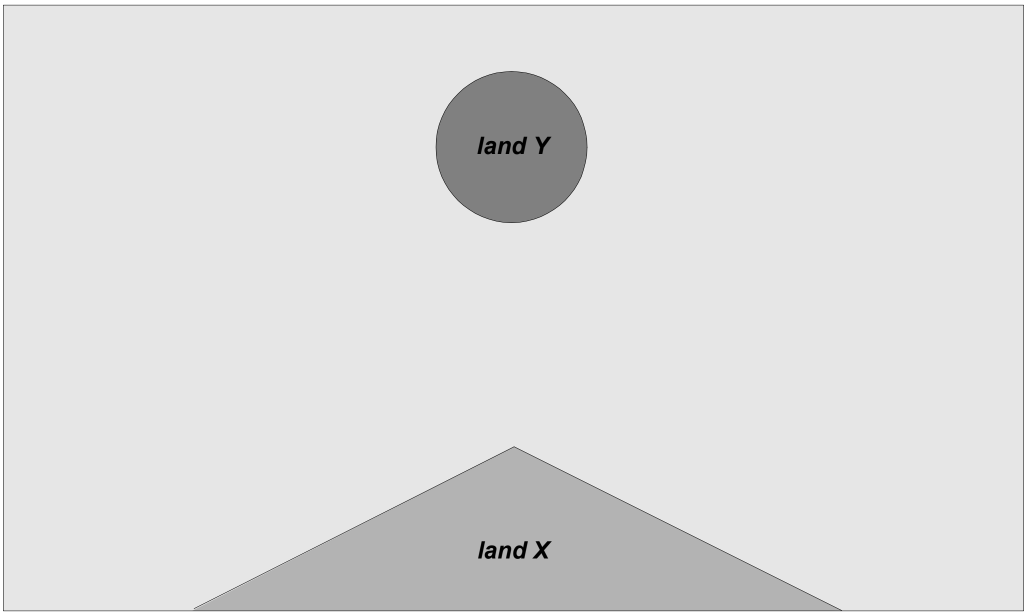


a. Draw for each of these five points the feet on the edge of land *X* and also on the edge of island *Y*.
b. Investigate for each of the points whether it lies closer to land *X* or to land *Y*.
c. Sketch how the conflict line runs approximately between the two lands. Make sure that the points P_1 trough P_5 lie on the right side of the conflict line.



25 In this figure an iso-distance line for the island has been sketched. This line consists of circle arcs and straight line segments. The segments of the iso-distance line join to each other on the dotted lines. These dotted lines are of great importance for the conflict line.
a. The dotted lines divide the region round the island in six sectors.
Indicate for the sectors 1 through 5 which shape the conflict line has in that segment. First think about which case from the survey scheme you are dealing with.
b. Sector 6 lies 'behind' the island. Investigate whether the conflict line also continues in this sector.
c. Draw in the figure an (compared to exercise **24**) improved version of the conflict line.

extra exercise



26 In the situation above the conflict line between the two lands consists of three parts. Sketch this conflict line.
Use the techniques you have seen in this section for the exact description of the conflict line.
Add a clear argumentation.

7: The tangent line property of the parabola

In practical applications of parabolas, ellipses and hyperbolas the special *tangent line property* of these curves is often being used. You have *seen* that the perpendicular bisectors, which appeared in the constructions were tangent lines to the parabola, the ellipse and the hyperbola. In this section we will prove this and then directly use the properties of the tangent lines in important applications of the three figures. The most important application has to do with reflection.

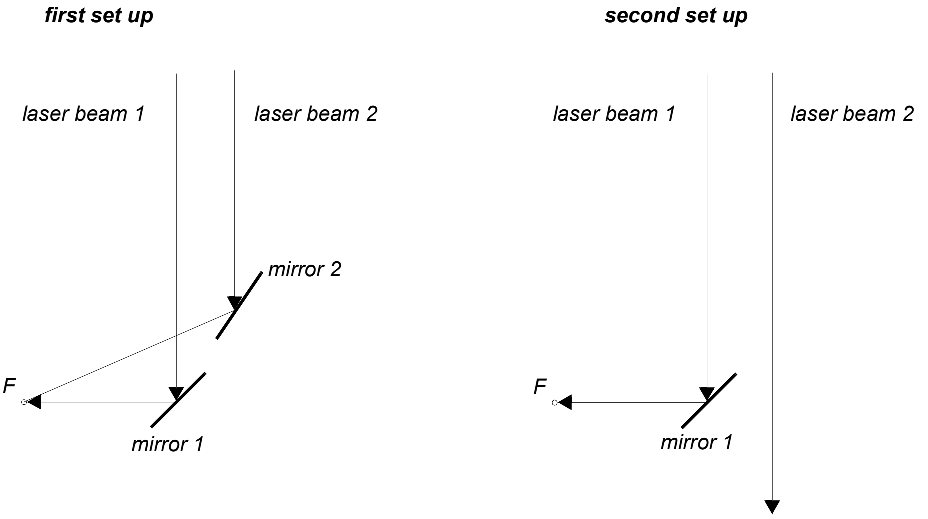
law of reflection

As known, the following *law of reflection* applies to plane mirrors:

angle of incidence = angle of reflection

If you have a plane mirror and you shine a parallel beam on it, a parallel reflected beam will reflect. This is not spectacular.
Our first goal is to design a mirror, *which converts a parallel beam into a converging beam*, i.e. into a beam which goes through one point.

In the left figure you see two mirrors and two laser beams. Laser beam 1 (2) is reflected via mirror 1 (2). Both rays reach point *F*.



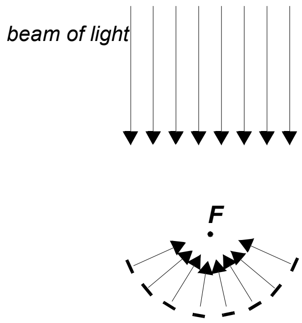
27 a. In the figure on the left, how can you check with your protractore that the reflection rays indeed go through *F*?

b. Draw in the right figure also a mirror 2 so that laser beam 2 is reflected on point *F* (on the place of the point of the arrow of laser beam 2).

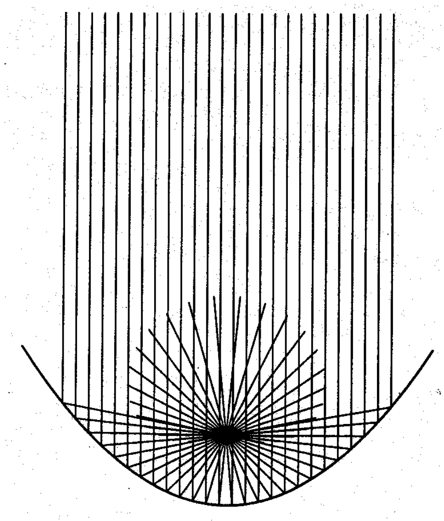
c. What do you notice when you compare the two positionings of mirror 2?

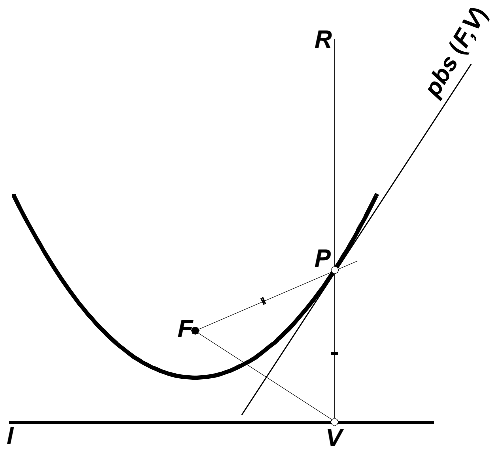
curved mirror

In principle you could reflect parallel laser beams using a hundred plane mirrors to one specific point. This is not very practical, for example the mirrors should not be in each other's way. With a *beam* of rays this is impossible. Then you would need infinitely many very small mirrors, which all together form a curved mirror. These need to ground in a certain shape. Also, for a curved mirror the law of reflection stands. With angle of incidence and angle of reflection we then mean the angles with the *tangent line* to the curve!



In the left figure below you see a mirror with the wanted curved shape: after reflection, all rays of light of the parallel beam converge to one point.
Next to that you see a sketch of a parabola as you have seen before.





A strong impression exists that the parabola is the wanted figure. A ray of light would approach from *R*, parallel to the axis of the parabola; the ray would hit the parabola at *P* and end up at *F* after reflection. Since this would be true for all rays parallel to the axis, the beam does converge in *F* after reflection.

First two things need to be checked:

I : the line *pbs*(*F, V*) is indeed the tangent line to the parabola

II : the lines *RP* and *PF* behave in relation to this perpendicular bisector according to the law of reflection: angle of incidence = angle of reflection.

28 The second point is the easiest to prove. Do that yourself first.

Next you are going to prove that the perpendicular bisector of *F* and *V* lies completely outside the parabola except for point *P*.

Given:	***P* is a point of the parabola with focus *F* and directrix *l*.** ***V* is the foot of *P* on *l*.**
To proof:	***pbs*(*F* , *V*) lies outside the parabola except for *P*.**

29 Choose an arbitrary point Q on pbs (F, V), other than P.
Now proof that Q lies outside the parabola, i.e. that $d(Q, F) > d(Q, l)$.
Hint: draw the line segment that realizes the distance d(Q, l).

a consideration in the margin

From this proof it follows that $pbs(F, V)$ lies outside the parabola except for P. Is it therefore also the tangent line? is the question now. Since the parabola is a smooth figure, it is hard to avoid that impression. But you actually should prove that there exists one and only one such line, which lies outside the parabola except for point P. Proving that this is the case would go too far at this moment. We will revisit this in chapter 3.
There it will be shown:
- that the graph of $y = x^2$ is legitimately indicated as 'parabola' and thus that you can find a focus F and a directrix l;
- that the graph of a linear approximation of $y = x^2$ in a point P *coincides with the perpendicular bisector* $pbs(F, V)$, where V again is the foot is of the perpendicular line from P on l.

Said otherwise: the tangent line as found in this chapter is the same as the one in differential calculus.

you can assume from now on that $pbs(F, V)$ is a tangent line to the parabola. In short: you can use the assertions made in I and II on the previous page from now on.

Everything about the *tangent line property of a parabola* is summarized in the next theorem:

tangent line-property of the parabola

> ***P* is a point on the parabola with focus *F* and directrix *l*.**
> **The tangent line in *P* to the parabola makes equal angles with line *PF* and the perpendicular line through *P* on *l*.**

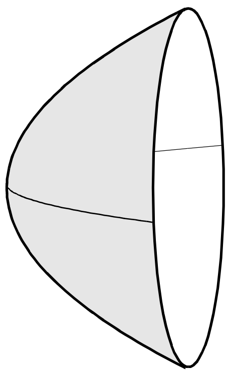
The physical meaning of the preceding is:

All rays of light, which approach a parabolic mirror parallel to the axis, will be reflected in the direction of the focus of the parabola.

30 Fill up yourself:
If rays ***go out*** *from the focus F, then the rays reflected by the parabolic mirror form a ....*

paraboloid

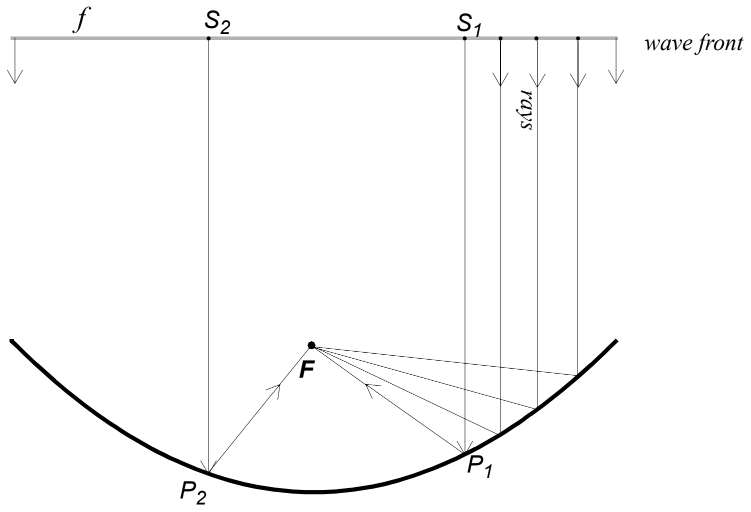
The headlight of a bike has the shape of a *paraboloid*.
A paraboloid is a spatial shape which arises from rotating a parabola about its axis of symmetry.



31 What can you say about the beam of light from such a bike lamp:
- **a.** if the filament of the light is exactly located in the focus?
- **b.** if the filament of the light is located a bit behind the focus?

parallel wave fronts

Dish aerials and radio telescopes also have the shape of a paraboloid (see the picture on the initial page of this chapter).
When receiving radio or tv signals, it is of importance that all 'rays ' reach the focal point at the same moment; only then is optimal reception possible.
You could imagine that a wave front (line *f* in the figure below) consists of points, which all move at the same speed in the same direction, namely the direction of the arrows.



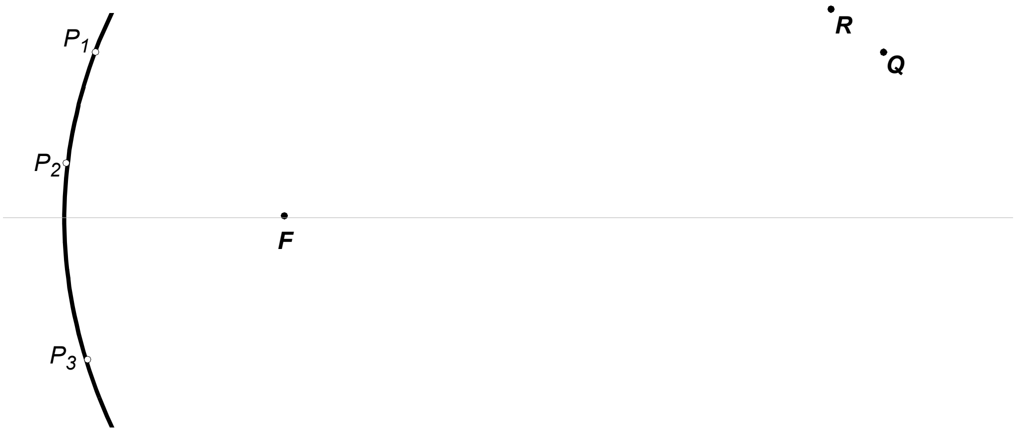
32 Explain that for a parabolic antenna all rays of light travel the same distance and that therefore the condition is met. Or: show that the paths S_1P_1F and S_2P_2F have the same length.

Nowadays you can see as many parabolic antennas as you want: satellite antennas are attached to many houses. They are aimed at satellites that have a fixed position above the equator. That means that in our regions the satellite antennas point south. In the city you do not need a compass to determine your orientation.

33 Can you name a few other applications of parabolic antennas?

construct oblique rays

34 Below, you can see a parabolic mirror; the axis of symmetry is indicated.



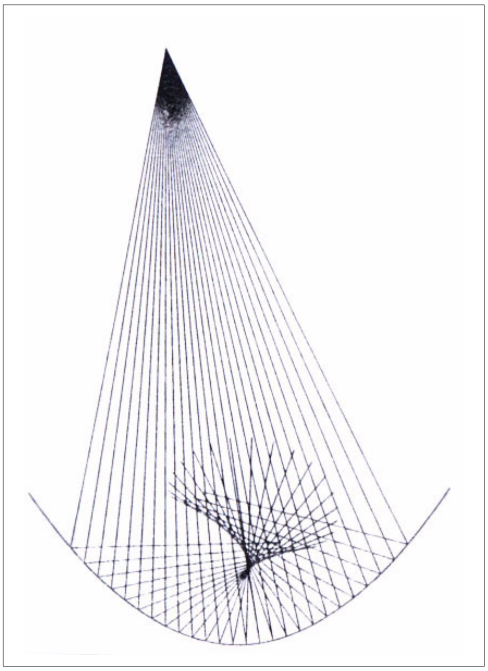
a. A ray, which falls from Q to P_1, is parallel to the axis. Draw the ray and its reflected continuation.

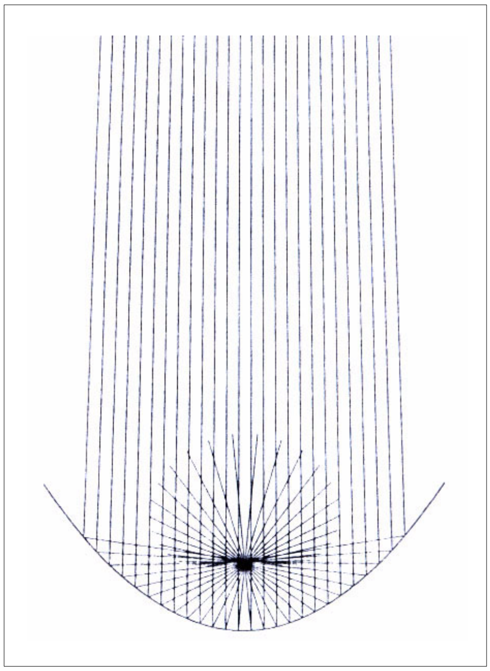
b. The ray, which falls from R on P_1, also answers to the law of reflection. Draw the ray and its reflected continuation. Now you need to sketch the tangent line in P_1 or its perpendicular line, i.e. the bisector of QP_1F. Use the protractor.

c. Also draw rays approaching from R to P_2 to P_3, and their reflected continuations.

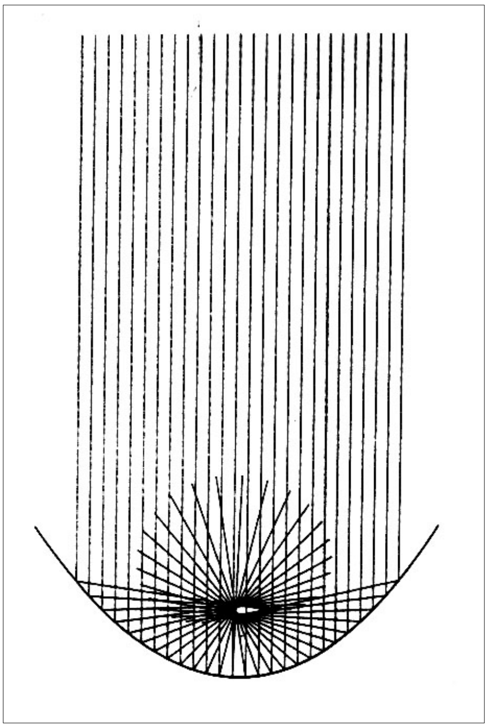
d. What do you notice about the rays approaching from R, and their reflected continuation?

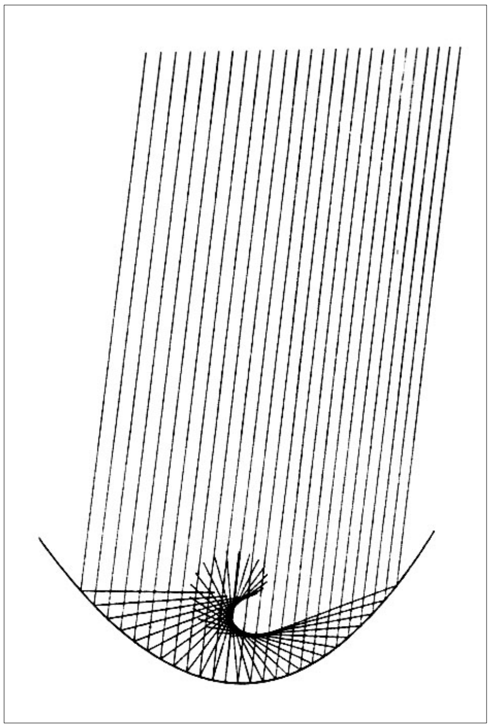
From the preceding you could draw the conclusion that non-parallel rays do not converge. In the figures below this can be seen again, but the right figure also shows: if the incident rays come from a point very far away, then convergence does occur in a reasonable approximation.





35 Below you see two cases where the beam is parallel, but not in the direction of the principal axis.





Give nuanced comments.

8: The tangent line properties of ellipse and hyperbola

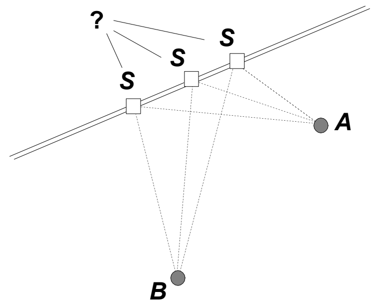
the case of the ellipse

While investigating the tangent line property of the ellipse we can unexpectedly reuse an old problem.

the reflection principle

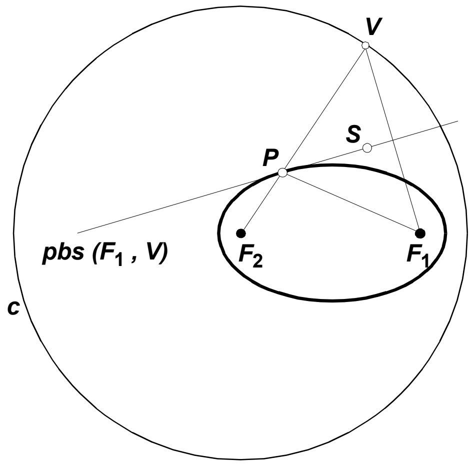
Maybe you recognize the following optimization problem.

**Cities *A* and *B* lie on the same side of a railroad.
One station needs to be built for both cities.
The bus company, which needs to bring the people from *A* and *B* either to or from this station, wants the position of the station to be chosen so that distance to the sum of the distances to *A* and *B* is as small as possible.**



36 Solve this problem again by applying the reflection principle.

The sketch on the right contains the same elements. *A* and *B* are now called F_1 and F_2. The optimal place of the station (point *P* in the figure) is the point where the line *l* (in the figure $pbs(F_1, V)$) touches the ellipse with the foci F_1 and F_2. *r* is here the radius of the director circle *c*.



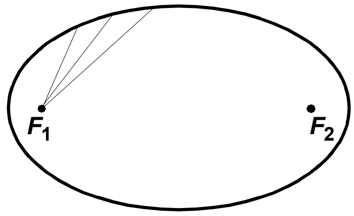
37 **a.** Prove that each other point *S* of the line *l* lies outside the sketched ellipse.
b. How follows from this that *P* is the wanted point?

38 Now prove the theorem about the tangent line property of ellipses.

***P* is a point on the ellipse with foci F_1 and F_2.
The tangent line in *P* to the ellipse makes equal angles with the lines PF_1 and PF_2**

(Here a hole in the proof will remain unplugged. Make the same assumption as we did for the parabola.)

39 Now apply the law of reflection to a concave elliptic mirror.
What happens to the rays of light which depart from focus F_1 and are reflected by the mirror?

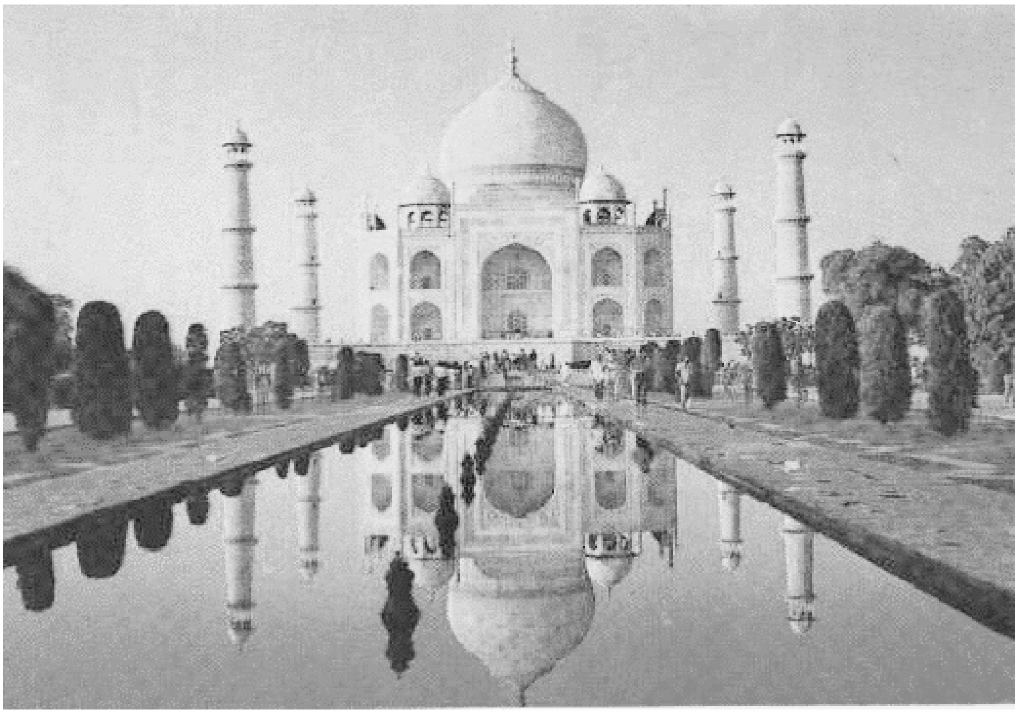


40 *At the dentist*.
Over the patient's head there is a bright lamp. The light of the lamp is reflected in a mirror and aimed at the oral cavity of the patient.

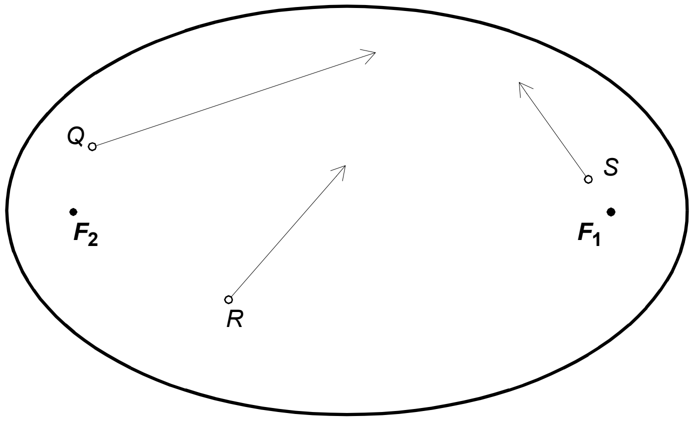
a. The dentist must perform his actions in a very small area, approximately 1 cm^2. Suppose that he wants the light concentrated here as much as possible, what shape should the mirror have?

b. Too large differences in light intensity between the work area and the direct surroundings are too tiring for the eye. Also, too much heat would be concentrated on an already ill molar. That is why the dentist would like a larger area to be lighted. How can that be achieved?

41 In the Taj Mahal in India a so-called *hall of private audience* can be found. Bridal couples who visited the Taj Mahal in the past had to stand on two special places in the hall of private audience, 15 meters apart. The groom whispered the vows of eternal love. His words were only heard by the bride.
Give comments, on love and ellipses.



42 Interesting is also the case of a ray of light within a reflecting ellipse that does not start in a focal point.



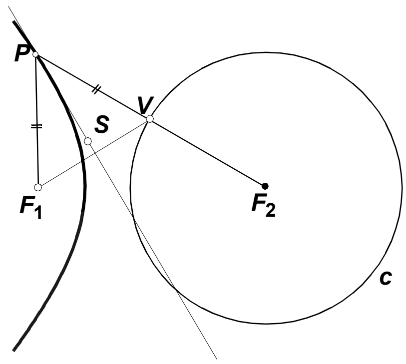
a. Continue with the ray starting at Q for a few reflections. Use the technique you have seen for the parabola in a modified way.

b. The ray from Q and all its reflections will never cross between the foci. Show this.

c. How is this for the rays starting at R and S?

the case of the hyperbola

Also in the figure of the hyperbola you see the same elements as for the parabola and ellipse.
The hyperbola branch lies, *except for point P,* again completely on one side of *pbs* (F_1, V).



43 For that must be shown:
$d(F_2, S) - d(F_1, S) < r$,
where r the radius of the circle is.
Give this proof by applying the triangle inequality to d(F_2, S). Find an appropriate triangle.

Thus here also applies:

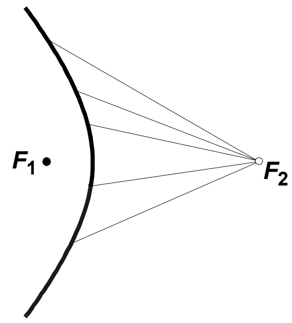
The perpendicular bisector pbs (F_1, V) *is the tangent line in P to the hyperbola branch.*

And the tangent line property also applies:

tangent line property hyperbola

> ***P*** **is a point on the hyperbola with foci** F_1 **and** F_2**.**
> **The tangent line in** ***P*** **to the hyperbola makes equal angles with the lines** PF_1 **and** PF_2**.**

44 On the right you see a *convex* hyperbolic mirror. F_1 and F_2 are the foci of the hyperbola.

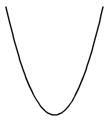
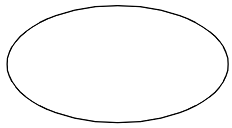
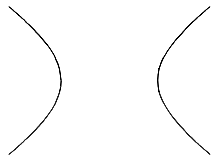


a. Find out in which direction the rays of light that depart from F_2 are being reflected.

b. Suppose, you have a *concave* hyperbolic mirror with a light source in F_1. Where do all rays of light appear to come from?

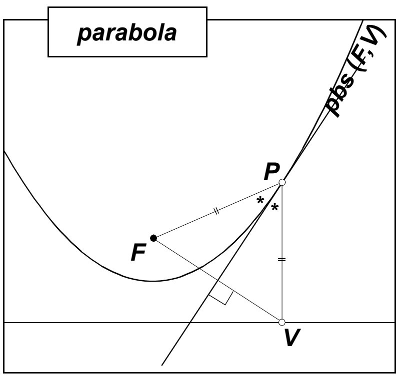
c. If in one way or another you have rays, which are all aimed at F_2, how will their reflection rays behave?

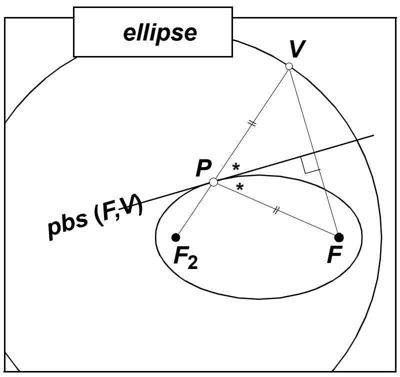
9: Survey parabola, ellipse and hyperbola

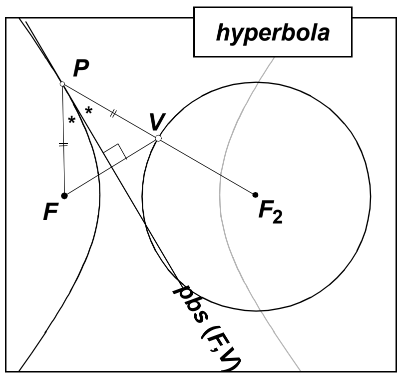
	parabola	**ellipse**	**hyperbola**
data	**Focus *F*, directrix *l*.**	**Foci F_1 and F_2, constant *r* ; $r > d(F_1, F_2)$.**	**Foci F_1 and F_2, constant *r* ; $r < d(F_1, F_2)$.**
condition for point P in the figure	**$d(P, F) = d(P, l)$.**	**$d(P, F_1) + d(P, F_2) = r$.**	**$\mid d(P, F_1) - d(P, F_2) \mid = r$.**
shape	**One branch, open on one side.** 	**Closed figure.** 	**Two open branches** 
number of vertices	**1**	**4**	**2**
symmetrical axes	**1**	**2**	**2**
directrices, circle directrices	**One director line.**	**Two director circles.**	**Two director circles.**
details	**All parabolas are similar.**	**The segments of the axes which lie inside the circle are called short and long axis.**	**One hyperbola has two asymptotes.**
tangent line property	**The tangent line in *P* to the parabola makes equal angles with line *PF* and the perpendicular line through *P* on *l*.**	**The tangent line in *P* to the ellipse makes equal angles with the lines PF_1 and PF_2.**	**The tangent line in *P* to the hyperbola makes equal angles with the lines PF_1 and PF_2**

Summarizing illustration to the tangent line properties

The naming in the following figures is the same everywhere:
- *P* is still a point on the figure (parabola, ellipse, hyperbola)
- *F* is the focus (or one of the foci)
- *V* is the foot point of *P* on the directrix or the director circle (if there are two, on the other focus).
  In all three cases the tangent line in P is the bisector of ∠ FPV and at the same time pbs(F, V)







10: (Extra) The folded path of light

You demand from a strong telephoto lens, which you photograph stars, sports or birds with, that:
- beams of parallel approaching light converge in one point
- that two beams, which hardly differ in direction (for instance two stars that can barely be told apart with the eye) still have convergence points which lie at a reasonable distance from each other.

If you want to accomplish this with ordinary lenses, you will end up with heavy lenses with long focal distances.
On the right you see a so called *catadioptric lens*. This is fairly compact and still has the desired properties.
Inside it contains two curved mirrors; the illustration on the next page shows a cross-section of the lens.
A parabolic mirror can be seen; actually, only part of it and it has a hole in the middle; its focus is indicated with F_1.
The ray of light approaching from S_1 runs parallel to the axis and thus reflects back in the direction of F_1. But a small curved mirror has been placed in front of F_1. This reflects the ray of light eventually to F_2, where the photographic film is located. (The shutter mechanism, which determines the exposure time, is not shown).

45 We further investigate the beam of rays.

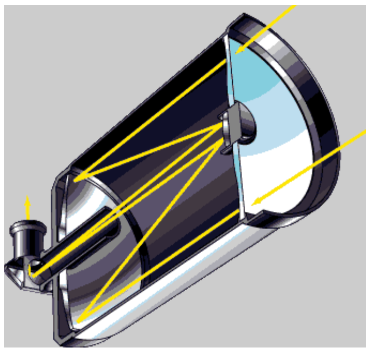
a. Finish the path of the rays approaching from S_2 and S_3. (These are parallel to the ray from S_1.) What shape should the little mirror have to make sure that these rays also end up in F_1?

b. The ray of light which approached from R_1, does not run parallel to the axis. Check that the ray does end up on point Q of the photographic film according to the law of reflection.

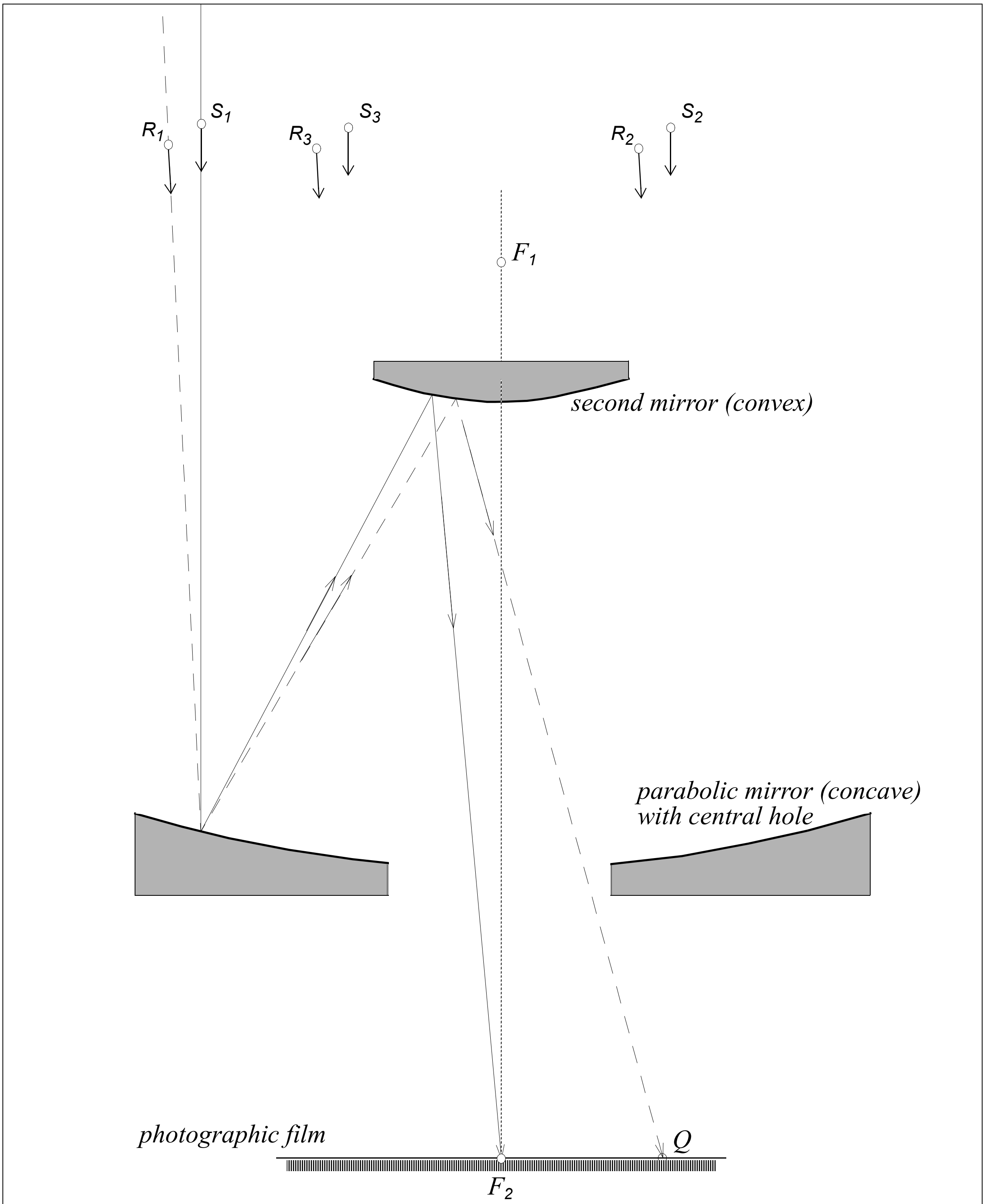
c. Construct the paths of the rays, which approach from R_2 and R_3 and are parallel to the one from R_1. Do all these arrive exactly in Q?

d. If the beam of the S- rays comes from one star and the one of the R-rays from another star close by, what happens then on the photographic plate?

e. Does the system meet the desired requirements?



The R-beam does not completely converge on the film. There are two solutions for this problem: make sure that the angle that the R-beam makes with the S-beam stays small
- construct a more complicated system, where a specially curved correction lens is used as well, as for example in the Schmidt-Cassegrain system, that among others is distributed by Cestron International. See the figure on the right.



beam of rays in catadioptric lens

Survey of Cabri

The most important features of CABRI

select

Put the pointer on the wanted object.
Click on the left mouse button.

drag

Put the pointer on the wanted object.
Click on the left mouse button, **move** the mouse with pushed button and let **go**.

Choice of bar, choice of option

Select a button and drag the mouse down, let go on the wanted option.
The active option remains visible in white in the bar.

POINTER MODE

By clicking on  you go into **pointer mode**. Hitting **ESC** also works.
In pointer mode you can move all kinds of things in the drawing.

draw and construct

Under the SKETCH and CONSTRUCT-buttons available **draw- and construct-options** are listed.
Choose the option and then click points or objects. This differs per option.
CABRI helps you if you want to use old points or lines if you want to attach things to them.
Always pay attention to the texts next to the pointer: the **tool tip**.
You can also use intersections of lines and circles.

leave Options

Leave options by going to **POINTER MODE**. This is, for example, necessary for the options EXTRA2>HIDE/SHOW and COLOUR.

Erase Objects AND SCREEN CLEANING

Select by clicking on an object in POINTER MODE; (with **SHIFT** +click you can select multiple items).
Hit the key **Delete** or **Back Space**.
With EDIT>SELECT ALL you can select all and then delete all.

UNDO

Most of the time you can recover your last action with tool bar EDIT>UNDO.

Dress up drawings

Under EXTRA1, 2 you find options for naming, coloring, line thickness, hiding objects, etcetera.

HELP

Use HELP and hope that you get enough information. CABRI in Help-mode supplies information about the chosen option. Also, the key **F1** turns HELP on and off.

Dynamics with CABRI

Move

If you drag a point or object, you can only do this if:

–it is a moving point on another object,

–it is an independent point.

What is dependent, moves accordingly.

Execution: Work from Pointer mode. Drag the to be moved object.

Animations

Works best for points which move over lines, segments and circles.

Use the option:

EXTRA1>ANIMATE

Drag the to be moved point and let it go.

It works like a pinball machine. The further you pull out the spring, the faster the movement is. It only works for independent objects.

Stop Animation

Hitting ESC or choosing anything with the mouse **stops** the animation.

Go back to the state before the movement with EDIT> UNDO .

You will still be in the Animate-option.

Turning on and Off trace

Choose the option:

EXTRA1>TRACE ON/OFF

Click on the object that should leave a trace.

Idem if you want to turn off the trace.

You can also leave traces of multiple objects at the same time.

Erase traces

Use the option:

EDIT>REFRESH DRAWING

Put the pointer on the wanted object.

Click on the left mouse button, **drag** the mouse with pushed in button, and **let go**.

Drawing the Path of moving point instantly

Choose the option:

CONSTRUCT1>LOCUS

After that, click:

first on the to be traced point (or the to be traced line)

and after that on the activating point.

Remark:

The path itself again is a new dependent object that can move and be distorted!

Adjust preferences for Locus

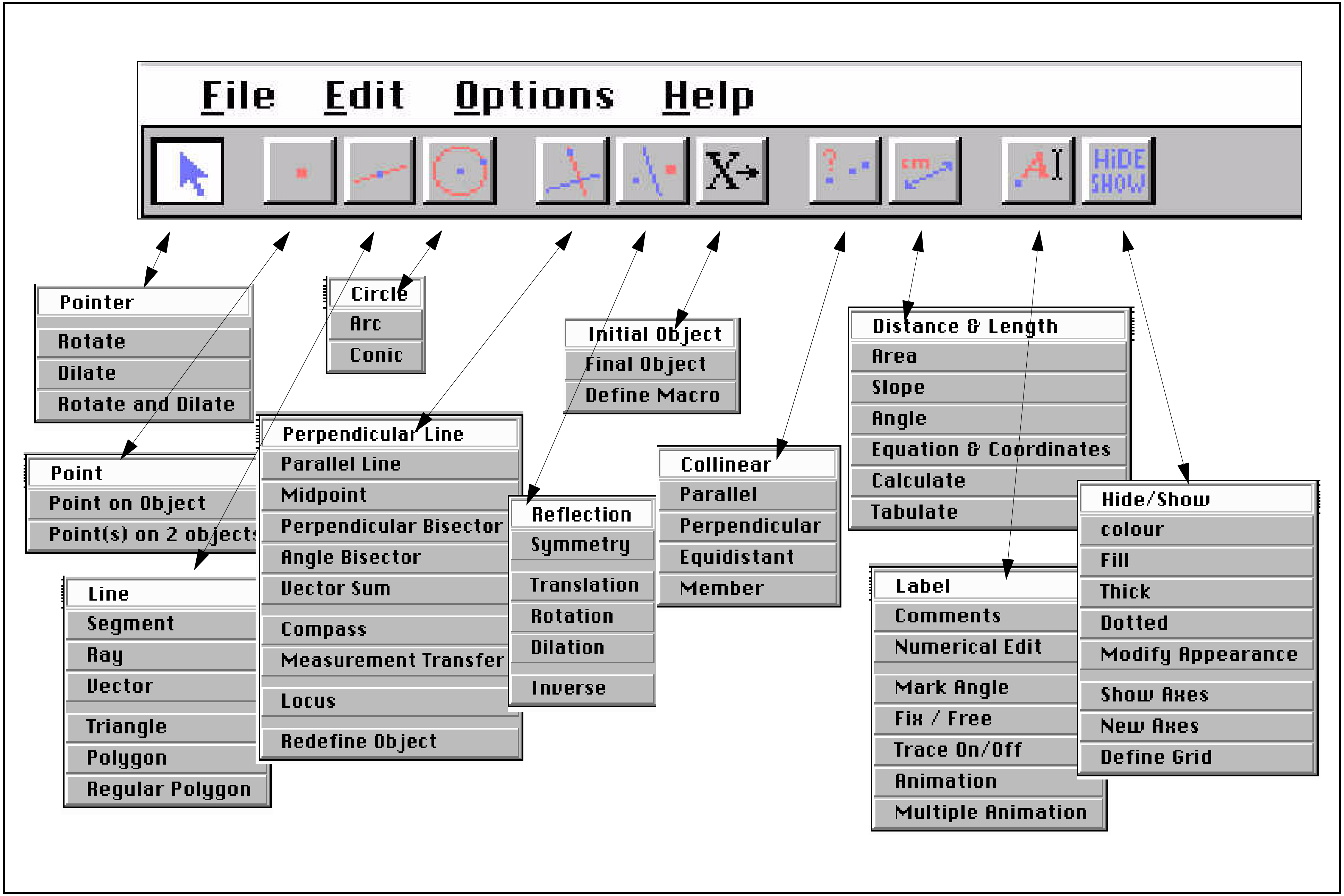
Via:

OPTIONS>PREFERENCES

Important choices:

–number of points for the locus

–Surrounding (Envelope) of the lines themselves for Locus-Or-Lines.


File
Edit
Options
Help
HIDE SHOW
Pointer
Rotate
Dilate
Rotate and Dilate
Point
Point on Object
Point(s) on 2 objects
Line
Segment
Ray
Vector
Triangle
Polygon
Regular Polygon
Circle
Arc
Conic
Perpendicular Line
Parallel Line
Midpoint
Perpendicular Bisector
Angle Bisector
Vector Sum
Compass
Measurement Transfer
Locus
Redefine Object
Reflection
Symmetry
Translation
Rotation
Dilation
Inverse
Initial Object
Final Object
Define Macro
Collinear
Parallel
Perpendicular
Equidistant
Member
Distance & Length
Area
Slope
Angle
Equation & Coordinates
Calculate
Tabulate
Label
Comments
Numerical Edit
Mark Angle
Fix / Free
Trace On/Off
Animation
Multiple Animation
Hide/Show
colour
Fill
Thick
Dotted
Modify Appearance
Show Axes
New Axes
Define Grid